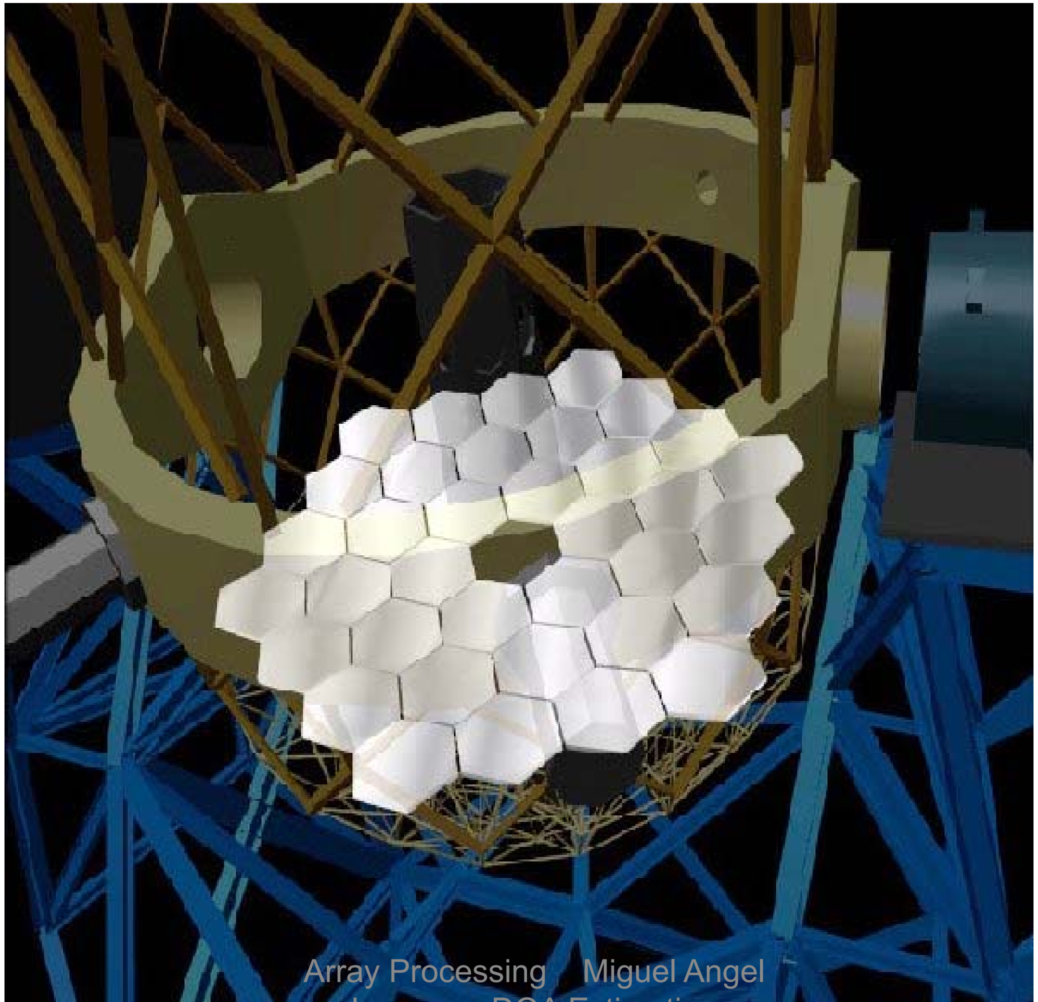
DIRECTION OF ARRIVAL ESTIMATION



Introduction

- Finding the direction of arrival of a wavefront is identical to the problem of finding power concentration in the frequency domain. In consequence….
- DOA estimation is the root problem of spectral density estimation dBm/Hz or just dBm/degree of solid angle.
- Differences: 3D search, non uniform sampling, wave propagation effects, no rational models, robutsness to missmatched is the must of DOA estimation methods.
- The problem: Giving a set of N snapshots from an aperture of Q sensors, to estimate the number of sources and their angles of arrival. When N> 10.Q the covariance is almost stabiliced.
- We will start with the narrowband problem for uncoherent point sources in the far field. The effect of coherent sources will be mentioned for each procedures and, finally, the wideband case will be presented.

The narrowband snapshot and covariance

$$\underline{X}_n = \sum_{i=1}^{NS} a_i(n).\underline{S}_s + \underline{w}_n = \underline{\underline{S}}_s.\underline{a}_n + \underline{w}_n$$

$$\underline{\underline{R}} = \underline{\underline{S}}_s.\underline{\underline{P}}_s.\underline{\underline{S}}_s^H + \underline{\underline{R}}_o = \sum_{r=1}^{NS} P(r).\underline{S}_r.\underline{S}_r^H + \underline{\underline{R}}_o$$

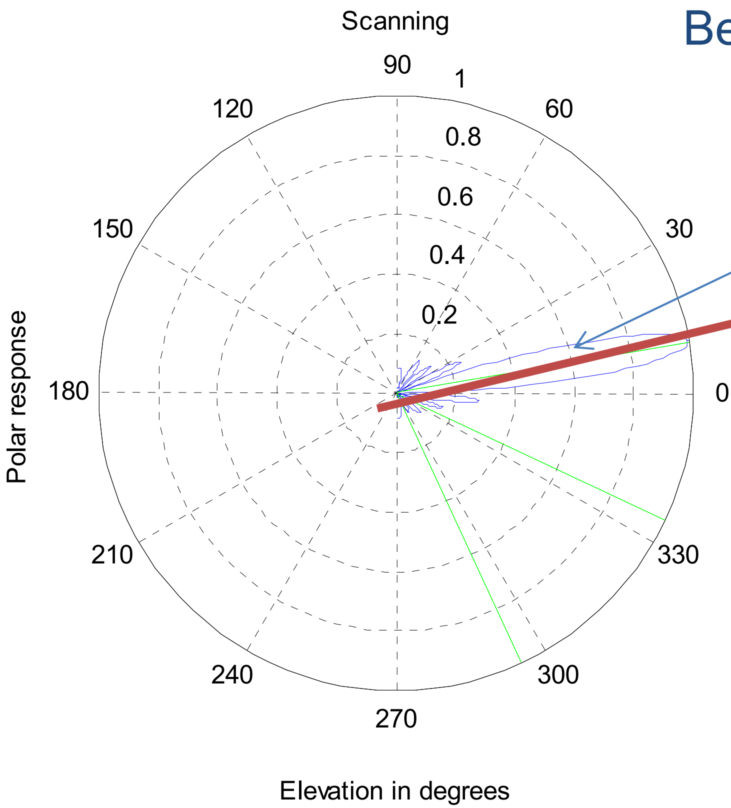
Our goal: To estimate the steering vector contained in the matrix of DOAs

Waveform and power of the sopurces is not our priority concern

Noise motivates the problem and converts a deterministic well defined problem in an estimation problem

To families of procedures: SCANNING METHODS
NULLING PROCEDURES (SUPERRESOLUTION)
ML BASED METHODS (OPTIMUM)

SCANNING Framework



Beamformer Response (to be designed A)

Steering Vector S
(Direcyion to scan)

In order to steer the beam to the desired direction a constraint is mandatory in the design

$$\underline{A}^H.\underline{S}=1$$

Just in case others directions should be nulled or attenuated (those where known stations or RF equipment is located. In this case te constrains would be NC<Q

$$\underline{A}^H.\underline{\underline{C}}^H=\underline{f}^H$$

Ince a beamformer A have been selected, we measure the power at the output of the beamformer as:

$$P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A} \quad mW.$$

The estimate is formed by the quotient of the power measured divided by the beamwidth of the beamformer.

TWO alternatives to derive a close expression of the effective beamwidth:

- From the equation that relates the output power with the beamformer directivity and the incoming power density.
- From a calllibration problem.

$$P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A} = \int_{\text{Field of View}} \left| \mathrm{A}(\underline{\theta}, \underline{\theta}_d) \right|^2 s(\underline{\theta}) d\underline{\theta}$$

Beamformer directivty, steered on the desired direction

Power density, i.e. the value to be estimated by any DOA procedure

Assuming that the beam is narrow with beamwidth Bw around the steered angle……

$$P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A} = \Phi(\underline{\theta}) \boxed{\int_{\text{Field of View}} \left| \mathrm{A}(\underline{\theta}, \underline{\theta}_d) \right|^2 d\underline{\theta}}$$

Estimate of s($\underline{\theta}$), due to the approximation, of the power density

$$\int_{\text{Field of View}} \left| \mathrm{A}(\underline{\theta}, \underline{\theta}_d) \right|^2 d\underline{\theta} = \underline{A}^H \underline{A}$$

In summary, given the beamformer $\underline{A}$ the power level estimate and the power density (local maxima will provide DOA estimates will be:

$$\boxed{\begin{gathered} P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A} \\ \Phi(\underline{S}) = \frac{\underline{A}^H . \underline{\underline{R}} . \underline{A}}{\underline{A}^H \underline{A}} \end{gathered}}$$

The calliobration alternative consists on choosing the beamwidth Bw in such a way that with undirectional noise only present in the scenario , i.e. Covariance equal to $\sigma^2.\underline{I}$, the estimat e will be equal to σ^2

$$\Phi(\underline{S}) = \frac{\underline{A}^H.\underline{\underline{R}}.\underline{A}}{Bw} = \left| \underline{\underline{R}} = \sigma \underline{\underline{I}} \right| = \sigma$$

Clearly the callibration requires that $$Bw = \underline{A}^H.\underline{A}$$

This beamwidth can be defined as the bandwidth of an ideal "brick shape" beamformer such that both, the original and the ideal, produce the same output power when only non-directional noise is present in the scenario.

SCANNING procedures differ on the way the scanning beamfomer is designed

The Phased Array (PA) Method

The easiest procedure for beamforming is the so-called phased array method. Imagine that we desire to steer the beam toward the broadside, in this case all the entries of the beam are one resulting in an almost no-operation processing. In fact, the mathematical ground for this design is just to select the beamfomer which produces the maximum dot product with the steering vector of the direction to scan.

In summary, the beamformer is selected from a non-bias constraint in the scanning direction and minimum norm or response to the leakage produced by the non-directional noise

$$\underline{A}^H.\underline{S} = 1$$

$$\left.\underline{A}^H.\underline{A}\right|_{min}$$

The solution is an only-phase beamformer known as the phased array.

$$\underline{A} = \frac{\underline{S}}{\underline{S}^H.\underline{S}} = \frac{1}{Q}.\underline{S}$$

Using this beamformer on the power and the spectral density estimators we have:

$$P(\theta,\varphi) = P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A} = \frac{1}{Q^2} . \underline{S}^H . \underline{\underline{R}} . \underline{S}$$

$$\Phi(\theta,\varphi) = \Phi(\underline{S}) = \frac{\underline{A}^H . \underline{\underline{R}} . \underline{A}}{\underline{A}^H . \underline{A}} \boxed{= \frac{1}{Q} . \underline{S}^H . \underline{\underline{R}} . \underline{S}}$$

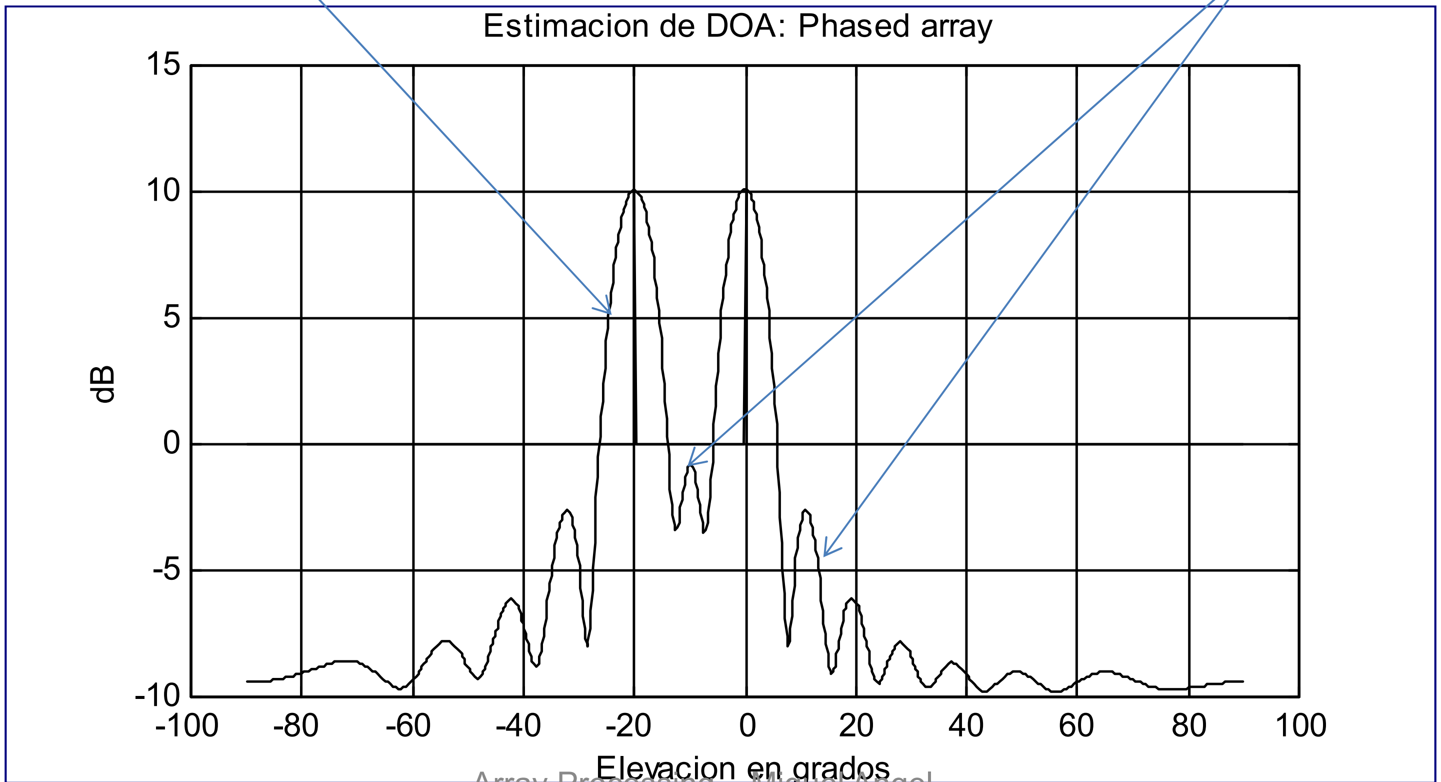
Note that the power level and the density level differ only in a constant. The reason for that is that this procedure is a constant-bandwidth scan since

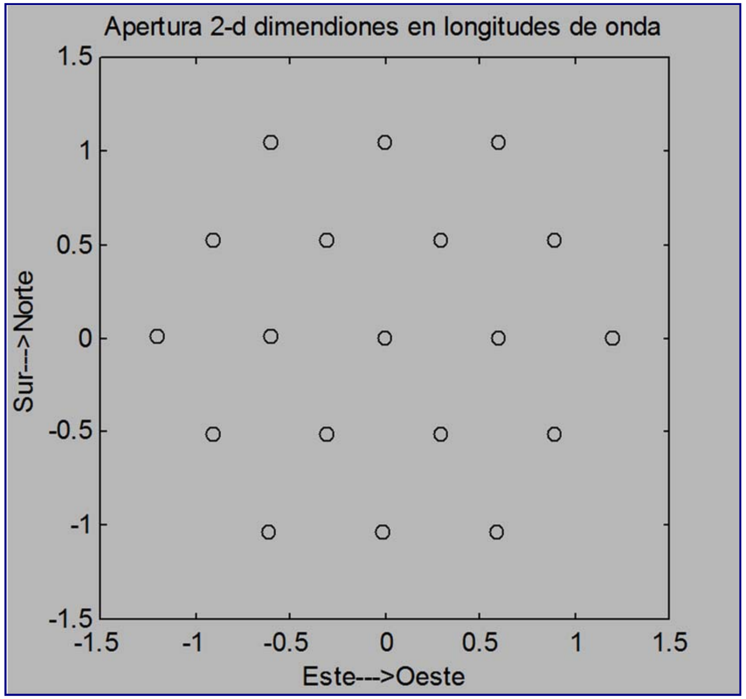
$$\underline{A}^H . \underline{A} = \frac{1}{Q}$$

PA estimates for an ULA array of 15 elements and using 3000 snapshots to compute the array covariance matrix R

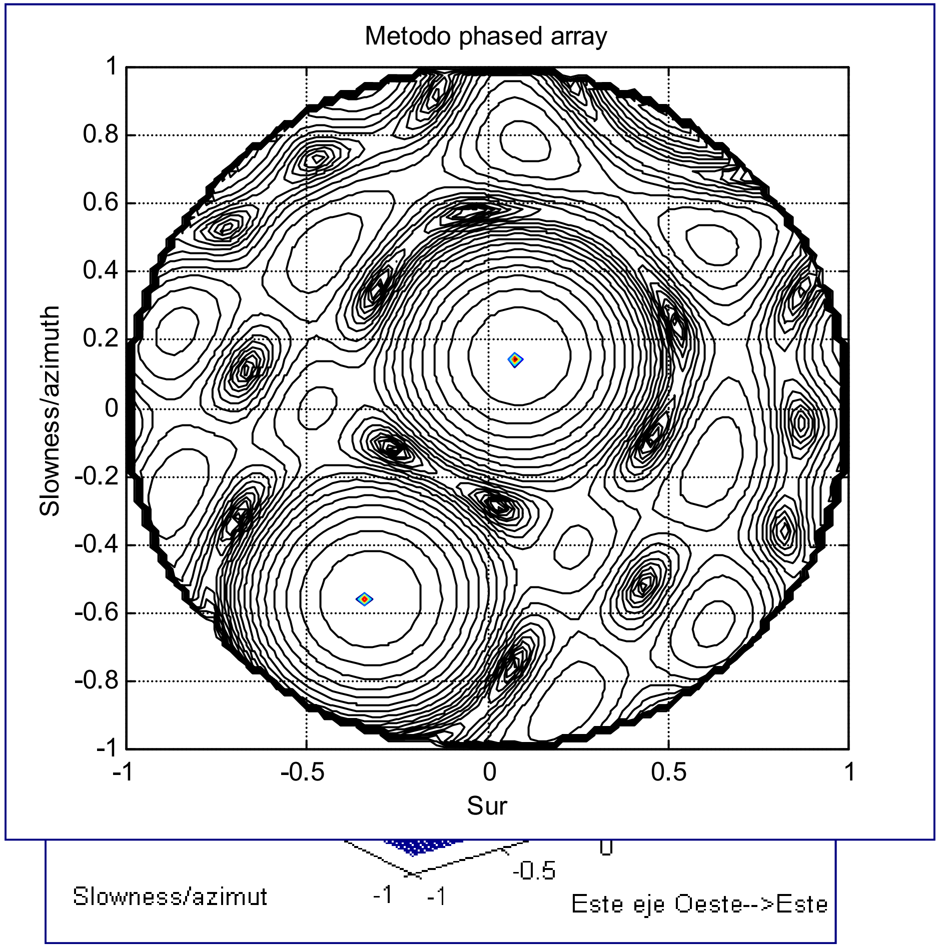
Resolution is limited by the aperture size (beamwidth of the scanning beam.

The major drawback of PA is the large leakage suffered on DOAs close where a source is present





For a planar aperture the performance is even worse since in somes directions the aperture is working down to 5 effective sensors instead of 15.



Data dependent beamforming

Since we desire to measure power impinging from the desired direction, power entering to the array from other directions will introduce a positive bias (leakage) on the power measured. In order to minimize leakage, the objective should be:

$$P(\underline{S}) = \underline{A}^H . \underline{\underline{R}} . \underline{A}\Big|_{MIN}$$

Knowing that the bias will be positive we have to minimize the output power.

In addition the constraint of 0 dB gain on the steering direction have to be added.

$$\underline{A}^H . \underline{S} = 1$$

Note that additional constrains can be added, i.e. null to engine noise source in towed arrays, known RF stations around, clutter, etc.

The MLM beamformer from Capon

The resulting beamformer is:

$$\underline{A} = \frac{\underline{\underline{R}}^{-1}.\underline{S}}{\underline{S}^{H}.\underline{\underline{R}}^{-1}.\underline{S}}$$

Note that this beamformer suffers from desired degradation when a coherent source is present on the scenario. The reason for coherent sources degrading the performance of MLM is that they promote negative bias on the power estimation, i.e. the beam steers the coherent source to minimize the output power when steered to the desired.

On mathematical terms, coherent sources degrade the rank of the covariance matrix of the sources degrading severely the performance of the method.

Regardless the method was denominated by Capon as maximum likelihood, this is not the case in general. ONLY when the beam is steered to one, and single, existing source in a non directional noise the power measured is MLM. Nevertheless, the optimum beamformer in this case coincides with a phased array

In order to do not produce any miss-understanding related with the original name of MLM, authors use to refer to this beamformer as minimum variance beamformer in the sense that it minimizes the variance at the beamforer output.

After using this beamformer for the power level and power density estimates we obtain:

$$\boxed{P(\underline{S}) = \frac{1}{\underline{S}^H . \underline{\underline{R}}^{-1} . \underline{S}} \; watts}$$

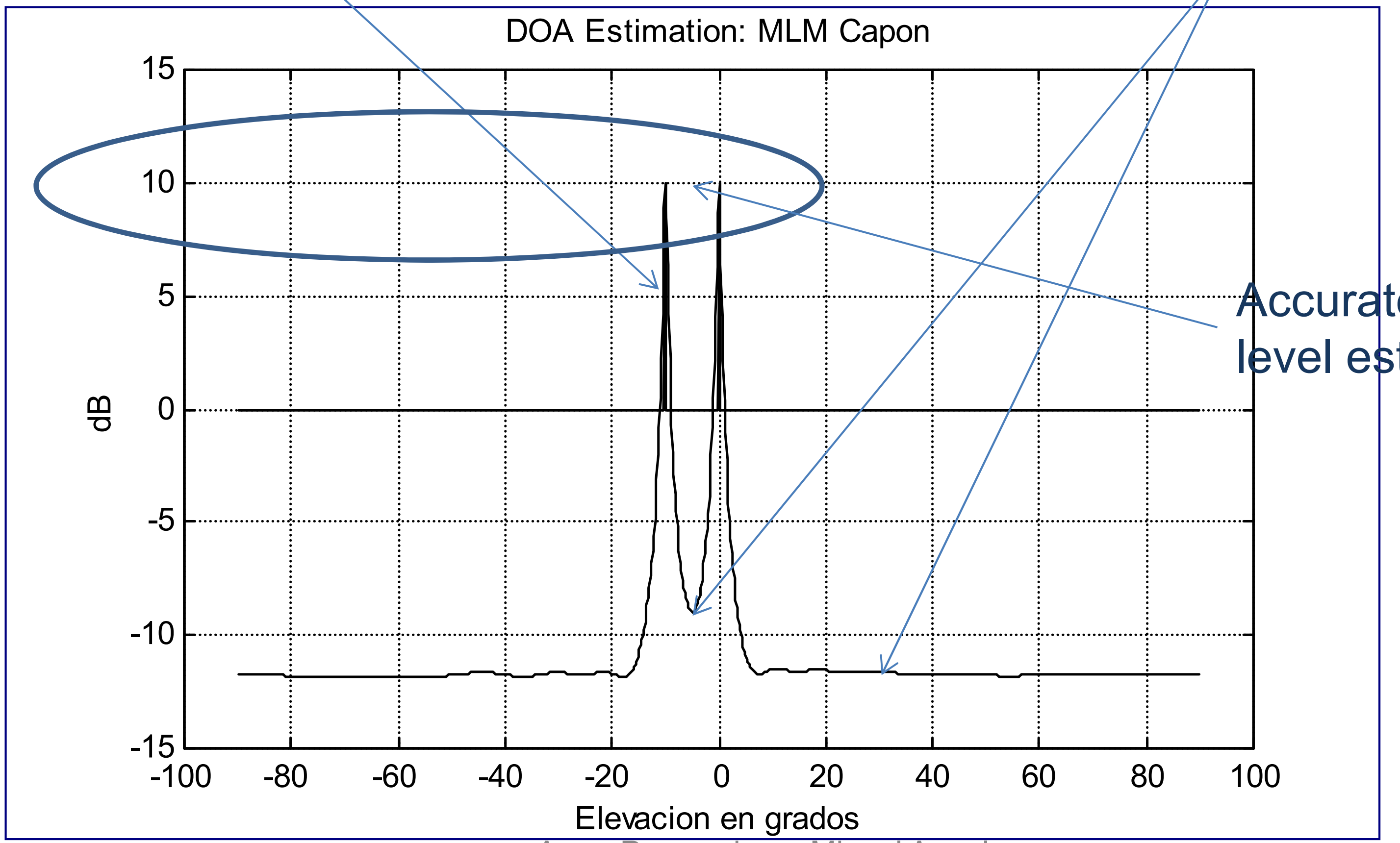
$$S(\underline{S}) = \frac{P(\underline{S})}{\underline{A}^H . \underline{A}} = \frac{\underline{A}^H . \underline{\underline{R}} . \underline{A}}{\underline{A}^H \underline{A}} = \boxed{\frac{\underline{S}^H . \underline{\underline{R}}^{-1} . \underline{S}}{\underline{S}^H . \underline{\underline{R}}^{-2} . \underline{S}}}$$

Scanning, as in all the procedures described herein is done by changing the steering angles within the steering vector <u>S</u>

Power level estimate MLM for an ULA array

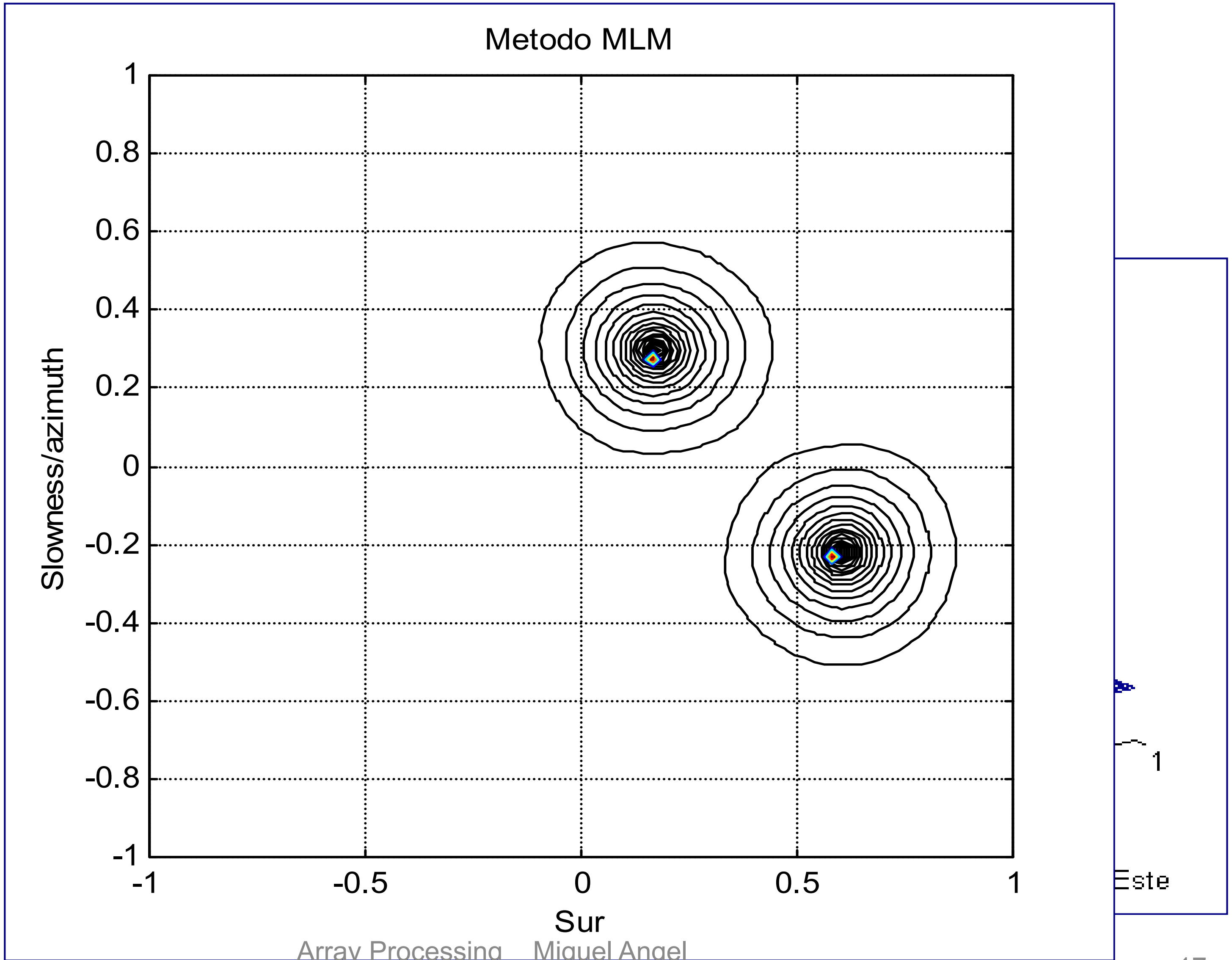
Better resolution than PA

Almost no leakage from existing sources

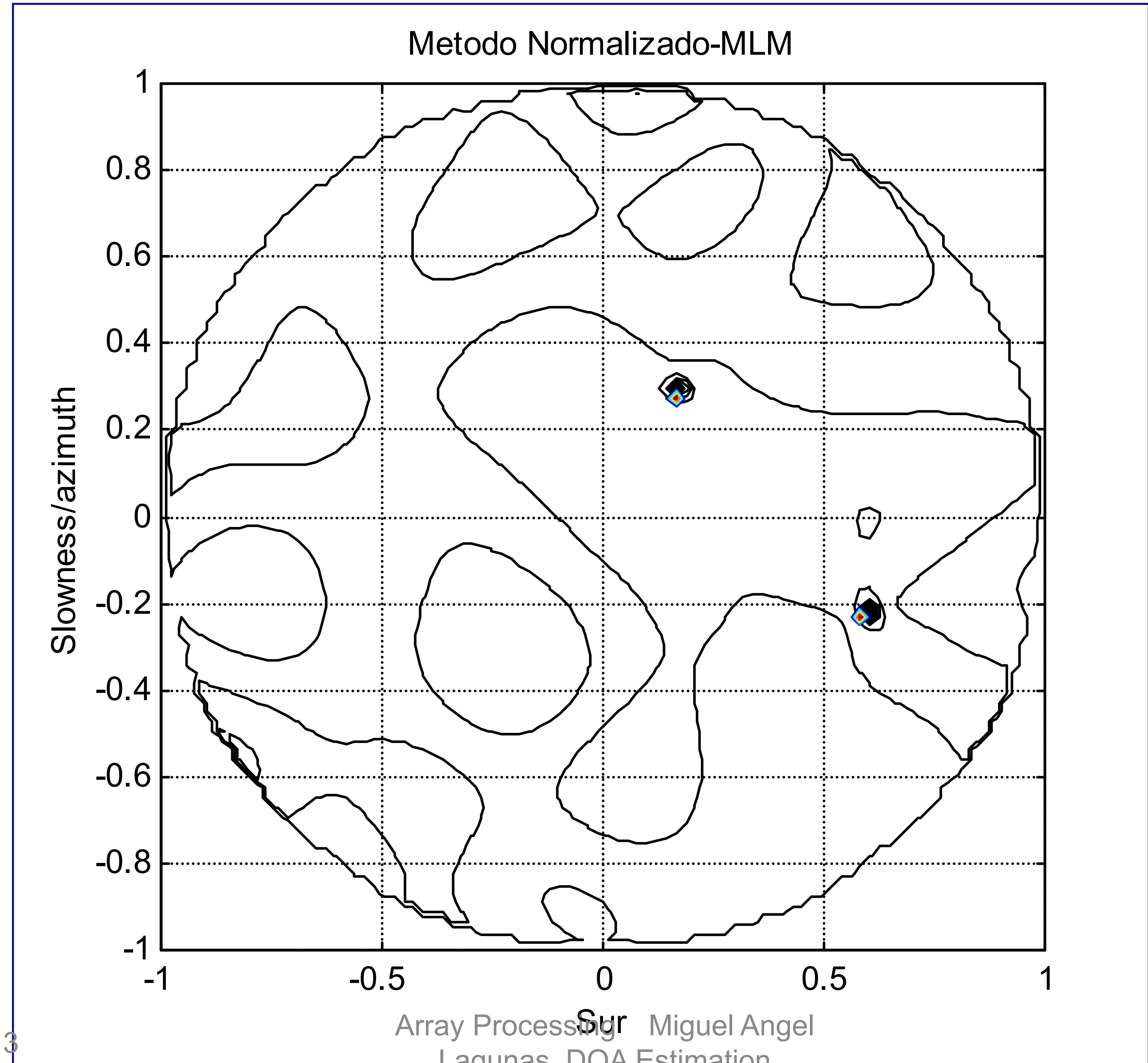


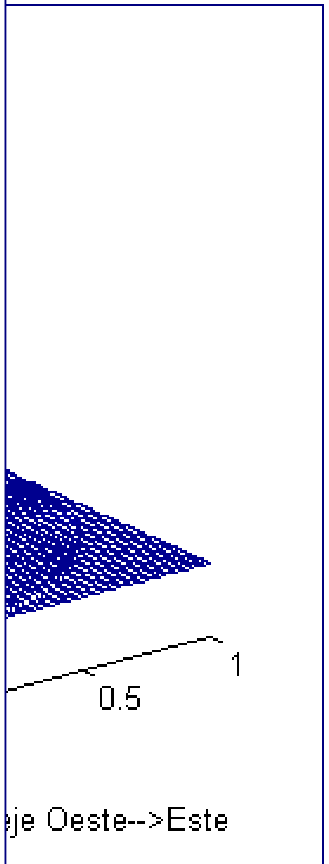
Accurate power level estimation

The improvement with respect PA is very evident for a planar aperture

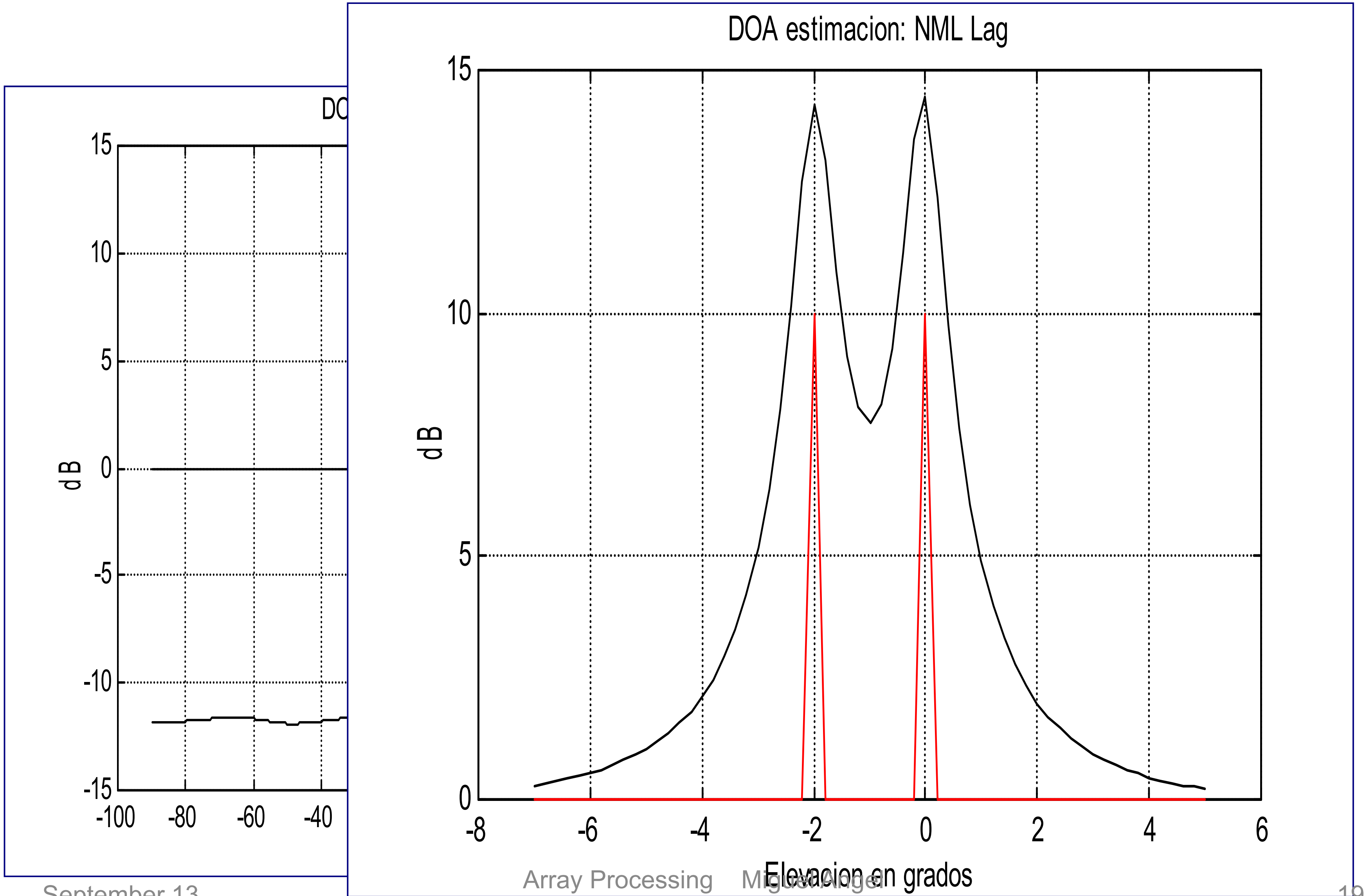


For the power density estimate the resolution further improves The leakage level is also improved. Of course power level is lost on density plots.





In order to show the performance of NMLM versus traditional MLM we need just to go to scenarios with closely spaced sources



Some authors, in order to use traditional MLM for density proposed (wrong) to use a constant bandwith 1/Q to produce a density estimate from the power estimate.

$$\frac{Q}{\underline{S}^H . \underline{\underline{R}}^{-1} . \underline{S}} \quad \text{watts/degree}$$

The formal proof of the superiority of NMLM versus MLM and PA, in terms of resolution we just need to use the fact that for density estimates we have:

$$\underline{u} = \underline{S}.\underline{\underline{R}}^{1/2} \quad \underline{v} = \underline{\underline{R}}^{-1/2}\underline{S} \quad \left(\underline{u}^H \underline{u}\right)\left(\underline{v}^H \underline{v}\right) \leq \left|\underline{u}^H \underline{v}\right|^2$$

$$\left(\underline{S}^H .\underline{\underline{R}}.\underline{S}\right)\left(\underline{S}^H .\underline{\underline{R}}^{-1}.\underline{S}\right) \geq Q^2 \Rightarrow \frac{Q}{\underline{S}^H .\underline{\underline{R}}^{-1}.\underline{\underline{S}}} \leq \frac{\underline{S}^H .\underline{\underline{R}}.\underline{S}}{Q}$$

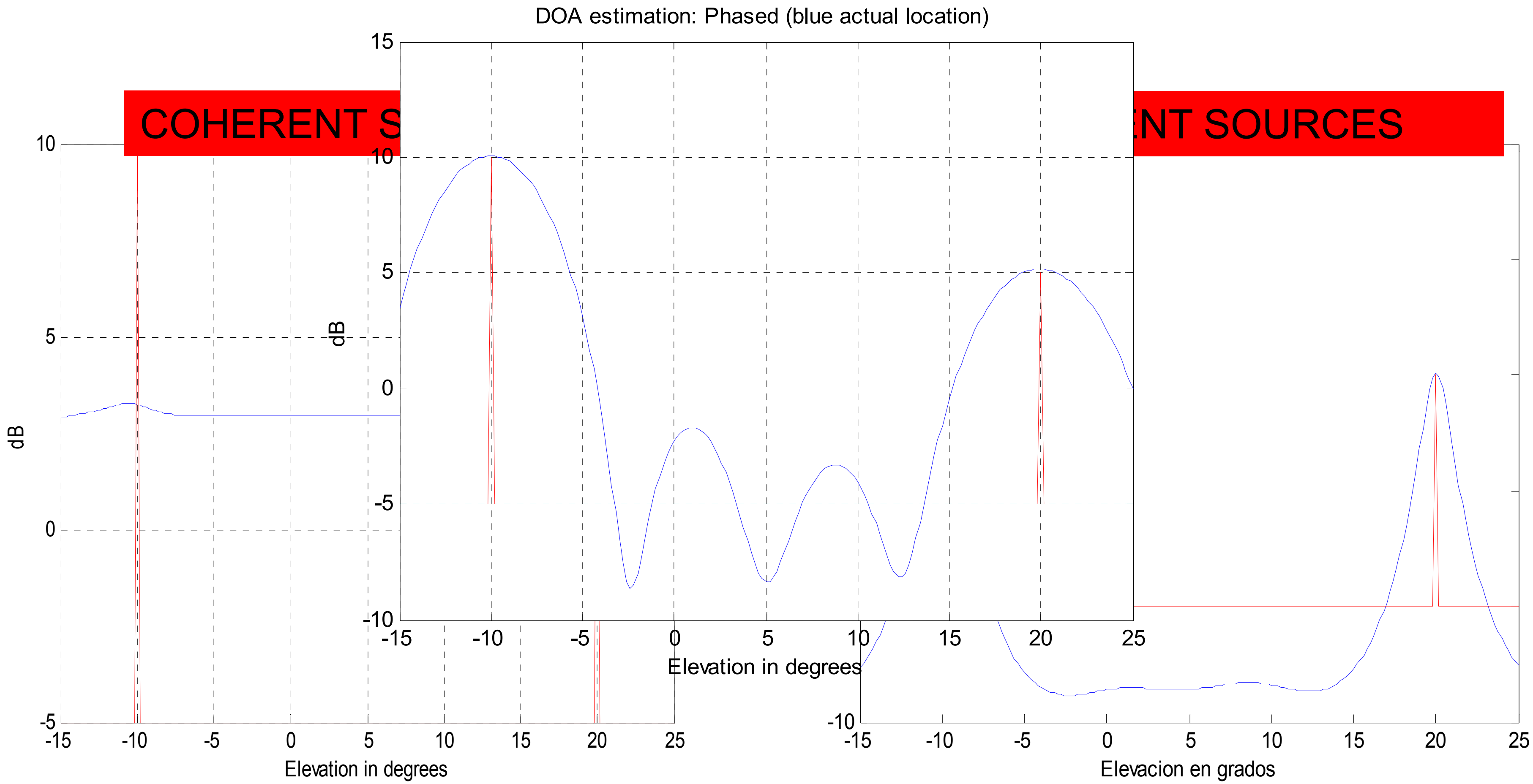
and

$$\underline{u} = \underline{S} \quad \underline{v} = \underline{\underline{R}}^{-1}\underline{S} \quad \left(\underline{u}^H \underline{u}\right)\left(\underline{v}^H \underline{v}\right) \leq \left|\underline{u}^H \underline{v}\right|^2$$

$$\left(\underline{S}^H .\underline{S}\right)\left(\underline{S}^H .\underline{\underline{R}}^{-2}.\underline{S}\right) \geq \left(\underline{S}^H .\underline{\underline{R}}^{-1}.\underline{S}\right)^2 \Rightarrow \frac{\underline{S}^H .\underline{\underline{R}}^{-1}.\underline{S}}{\underline{S}^H .\underline{\underline{R}}^{-2}.\underline{\underline{S}}} \leq \frac{Q}{\underline{S}^H .\underline{\underline{R}}^{-1}.\underline{S}}$$

Thus, assuming all get the same level at the source location, NMLM falls down faster than the other two methods.

Note the degradation of NMLM (and MLM) for coherent sources. It is worthwhile to remark that this effect is identical to the desired cancellation on SLC and GSLC beamformers.



Nulling procedures (Super-resolution methods)

Remarks:

- The size of the aperture bound the minimum beam-width of a beamformer response.
- Size do not preclude close located zeros of the beamforer response.
- The number of elements bounds the leakage suffered by a beamformer
- The number of elements (minus one) limits the number of zeros of the beamformer response.

IDEA:

Instead of looking for maximum power associate to the presence of sources, change the procedure such that minima (or zeros) are associated to the presence sources→ SUPER-RESOLUTION

HOW TO DO IT:

Design a beamformer $\underline{A}$ such that minimices the output power. To do so the objective is the same that in scanning methods.

Set some constraint to prevent the trivial solution, i.e. the zero beamformer.

Compute the response of the beamformer, since to do properly its job, it will present minima or zeros response to those DOA where a source is present in the scenario.

GENERAL FORMULATION:

Objective

$$\underline{A}^H . \underline{\underline{R}} . \underline{A} \Big|_{MIN}$$

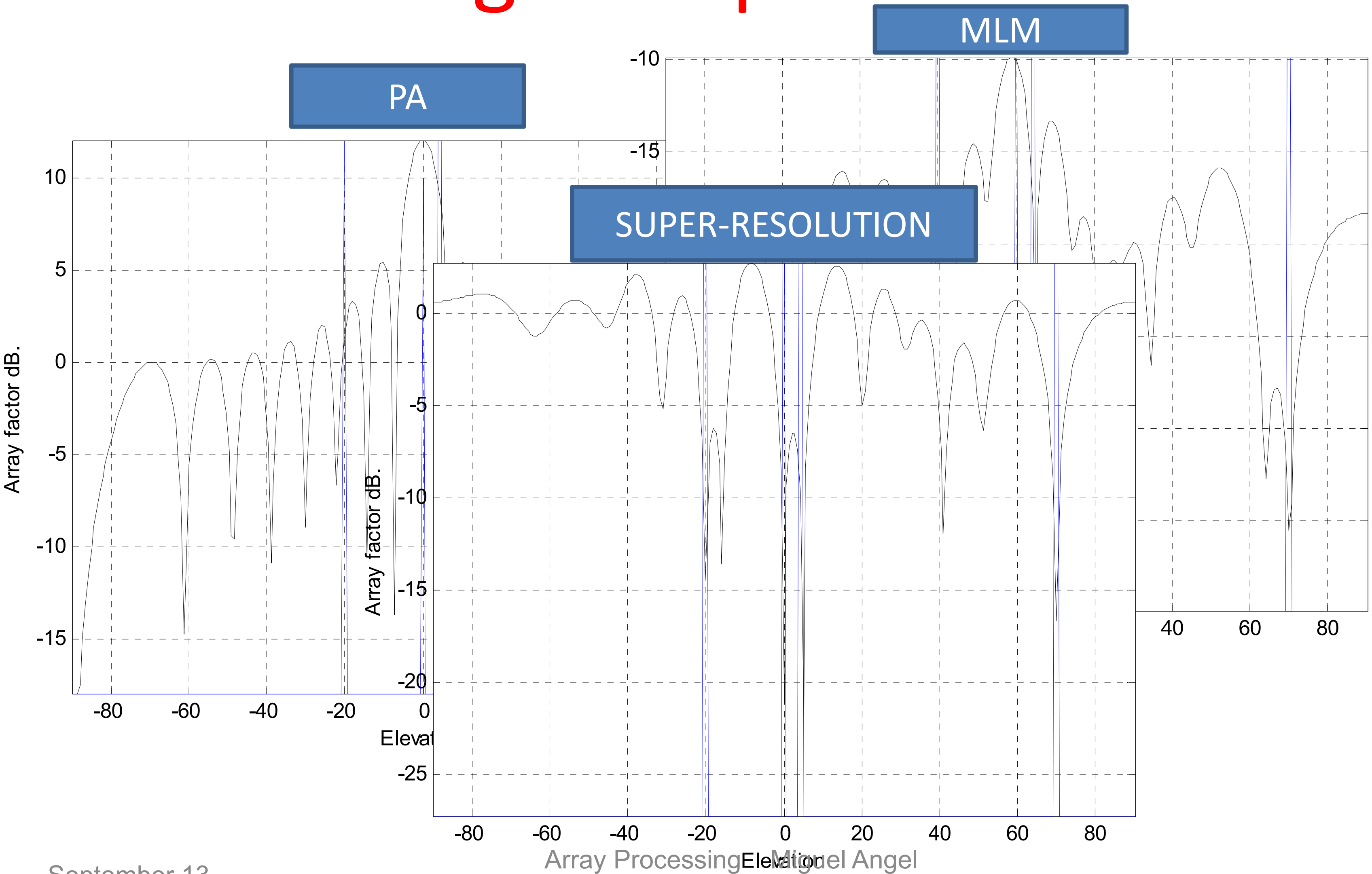
Constraint

$$f\left(\underline{A}\right) = c$$

Source detector as the maxima of the inverse of the beamfomer spatial response.

$$s\left(\underline{S}\right) = \frac{1}{\left|\underline{S}^H \underline{A}\right|^2}$$

Difference of beamformers for scanning or super-resolution



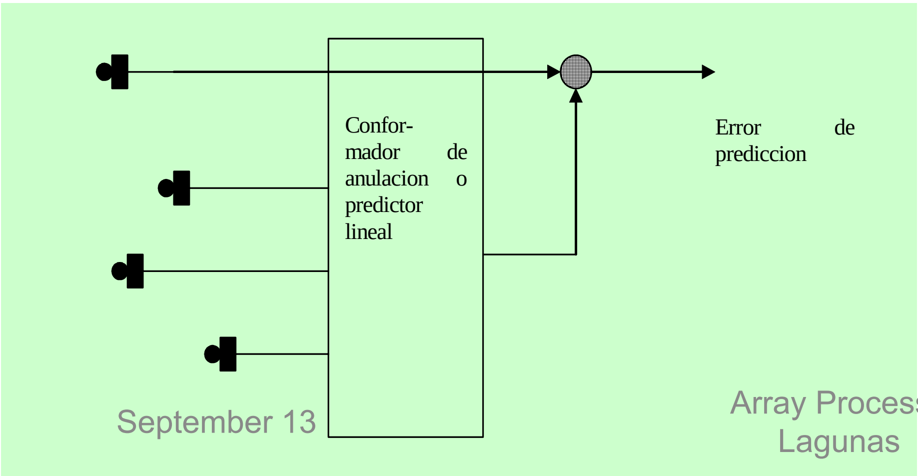
The spatial linear predictor

$$\underline{A}^H . \underline{\underline{R}} . \underline{A}\Big|_{MIN}$$

The constrain will be setting to one the beamformer weight of one element of the aperture.

$$\underline{A}^H \underline{1} = 1 \quad with \quad \underline{1} = \begin{bmatrix} 0 & .. & 0 & 1 & 0 & .. & 0 \end{bmatrix}$$

The solution for the beamformer is: $\underline{A} = \dfrac{\underline{\underline{R}}^{-1} . \underline{1}}{\underline{1}^H . \underline{\underline{R}}^{-1} . \underline{1}}$



And, the corresponding estimate is:

$$\boxed{\Phi(\underline{S}) = \frac{1}{\left| \underline{1}^H . \underline{\underline{R}}^{-1} . \underline{S} \right|^2}}$$

Pisarenko

$$\underline{A}^H.\underline{\underline{R}}.\underline{A}\Big|_{MIN}$$

The constrain will be setting to one the norm of the beamformer which is the same that asking for minimum response to the un-directional noise

$$\underline{A}^H\underline{A}=1$$

The solution for the beamformer is the Minimum eigenvector of this problem:

$$\underline{A}=\min eigenvector\ of \quad \underline{\underline{R}}.\underline{e}=\lambda.\underline{e}$$

And, the corresponding estimate is:

$$\boxed{\Phi(\underline{S})=\frac{1}{\left|\underline{e}_{\min}^H.\underline{S}\right|^2}}$$

COMMENTS:

- The estimates produce false peaks.
- The number of elements is greater than the number of existing sources.
- For Q elements the response may have up to Q-1 zeros, being NS the number of sources, it will be Q-1-NS false sources.
- Pisarenko and Prony solved the problem of labeling actual sources.
- S,R,K solved in an elegant manner this problem of resolution methods with the so-called MUSIC

MUSIC (Goniometre)

Super-resolution is valid for point sources and, in general for un-directional front-end noise. For this reason, people use to refer these procedures as point source detectors (far-field scenarios).

Under the above hypothesis, the array covariance matrix has a solid structure that can be used to improve the point source location performance.

$$\underline{\underline{R}} = \boxed{\sum_{s=1}^{NS} P_s.\underline{S}_s.\underline{S}_s^H} + \boxed{\sigma^2.\underline{\underline{I}}} = \underline{\underline{S}}.\underline{\underline{P}}.\underline{\underline{S}}^H + \sigma^2.\underline{\underline{I}}$$

SIGNAL SUB-SPACE
- Dimenssion NS, or rank NS, is formed from NS independent steering vectors

NOISE SUB-SPACE
- Full rank Q and formed by Q vectors orthogonal and equal to the columns of the identity matrix.

- The dimension of the array covariance is Q, thus the covariance matrix can be described from an orthonormal base of dimension Q.

- Within this space of dimension Q there is a subspace of lower dimension NS where the steering vectors we are interested for, form a non orthogonal base of it.

- Let us assume that vector $\underline{e}_q$ (q=1,Q) is one of the Q vectors describing the full space of the aperture, the power associated with this vector will be:

$$\underline{e}_q^H \underline{\underline{R}}.\underline{e}_q = \lambda_q$$

- It is obvious that those vectors that take part of the signal subspace will have more power associated than those which only, and just only, describe the noise-subspace.

- We assume that vectors form an orthogonal base, the noise vectors will be orthogonal to the vectors describing the signal subspace and in consequence THEY WILL BE ORTHOGONAL TO THE STEERING VECTORS OF THE SOURCES

The eigenvectors of the covariance matrix is the proper base to choose for our analysis.

$$\underline{\underline{R}} = \sum_{q=1}^{Q} \lambda_q . \underline{e}_q . \underline{e}_q^H = \underline{\underline{E}} . \underline{\underline{D}} . \underline{\underline{E}}^H$$

being

$$\underline{\underline{E}} = \begin{bmatrix} \underline{e}_1 & . & \underline{e}_{NS} & \underline{e}_{NS+1} & . & \underline{e}_Q \end{bmatrix}$$

the eigenvectors satisfy that

$$\underline{\underline{R}} . \underline{e}_q = \lambda_q . \underline{e}_q$$

Note that the eigenvalue is just the power associated with the corresponding eigenvector when we look at it as a beamformer

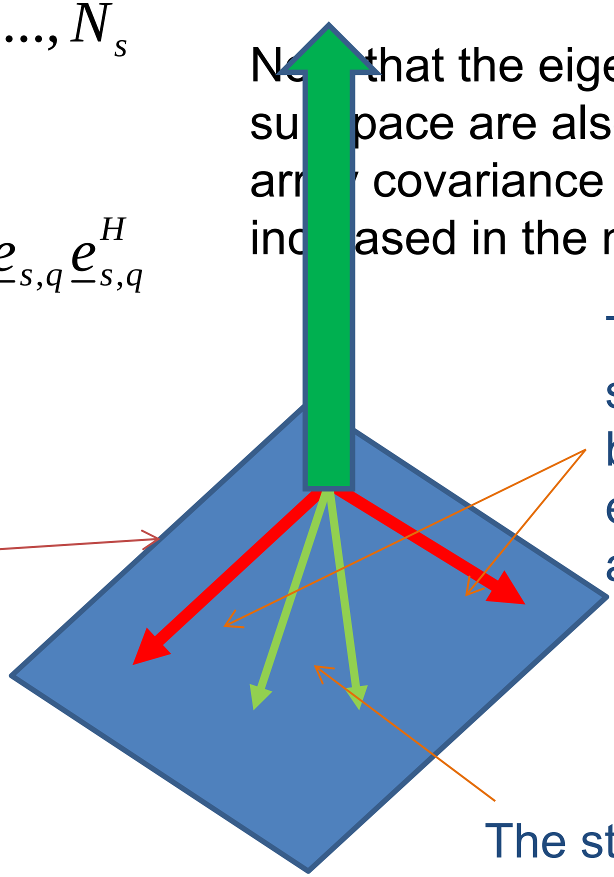
As the largest eigenvalues are associated with the point sources we may say that the corresponding NS eigenvectors fully describe the signal sub-space.

$$\underline{\underline{R}}_s \, \underline{e}_{s,q} = \lambda_{s,q} \underline{e}_{s,q} \qquad q = 1, \ldots, N_s$$

or

$$\underline{\underline{R}}_s = \sum_{q=1}^{Ns} \left| \alpha_q \right|^2 \underline{S}_q \underline{S}_q^H = \sum_{q=1}^{Ns} \lambda_{s,q} \underline{e}_{s,q} \underline{e}_{s,q}^H$$

Note that the eigenvectors of the signal subspace are also the eigenvector of the array covariance and the eigenvalue is increased in the noise level



The signal sub-space is described by the two largest eigenvectors which are orthogonal

Signal sub-space (Since Ns=2 it is a plane)

The steering vectors of the sources, not orthogonal, have to be in the plane.

THE NOISE EIGENVECTORS, ALL OF THEM WITH EIGENALUE EQUAL TO σ2, WILL BE ORTHOGONAL TO THE SOURCE STEERING VECTORS

MUSIC

- Compute eigenvectors and eigenvalues
- $\boxed{\text{Decide the dimension of the signal sub-space}}$
- Form and find the maxima of the MUSIC estimate

$$\Phi(\underline{S}) = \frac{1}{\sum_{q=NS+1}^{Q} \left| \underline{S}^{H} . \underline{e}_q \right|^2}$$

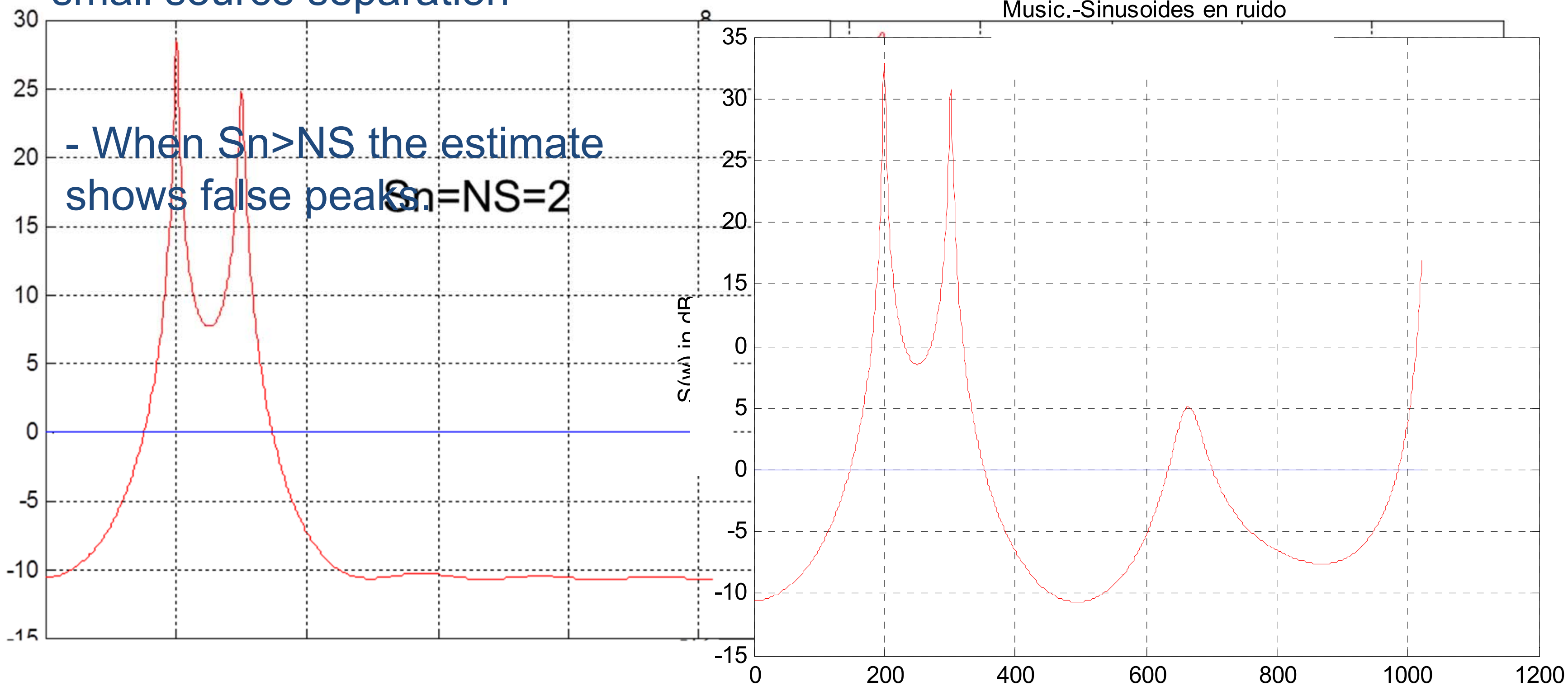
Note that every beamformer or eigenvector have Q-1 minima but only at the NS source location they will coincide. In this manner MUSIC removes the problem of false peaks. Note also that Pisarenko reduces to MUSIC when the dimension of the noise subspace is one, i.e. is identical when number of actual sources is Q-1.

The major problem of MUSIC, general for any subspace-based method, is to decide the length of the signal/noise subspace.

- When Sn (dimenssion)<NS (actual # of sources) the estimate tends to show less peaks than actual sources.

- When Sn=NS the result is excellent even for very small source separation

- When Sn>NS the estimate shows false peaks

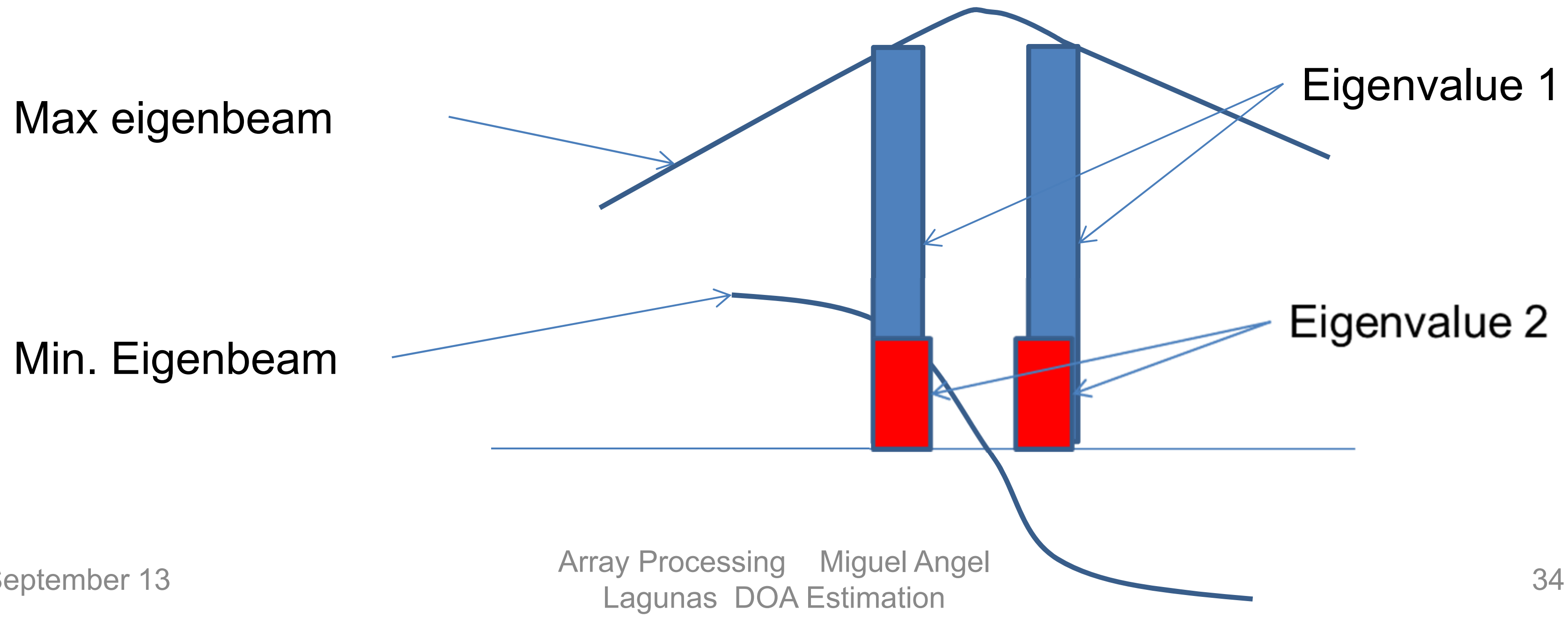


SELECTING THE DIMENSSION OF THE SUBSPACES

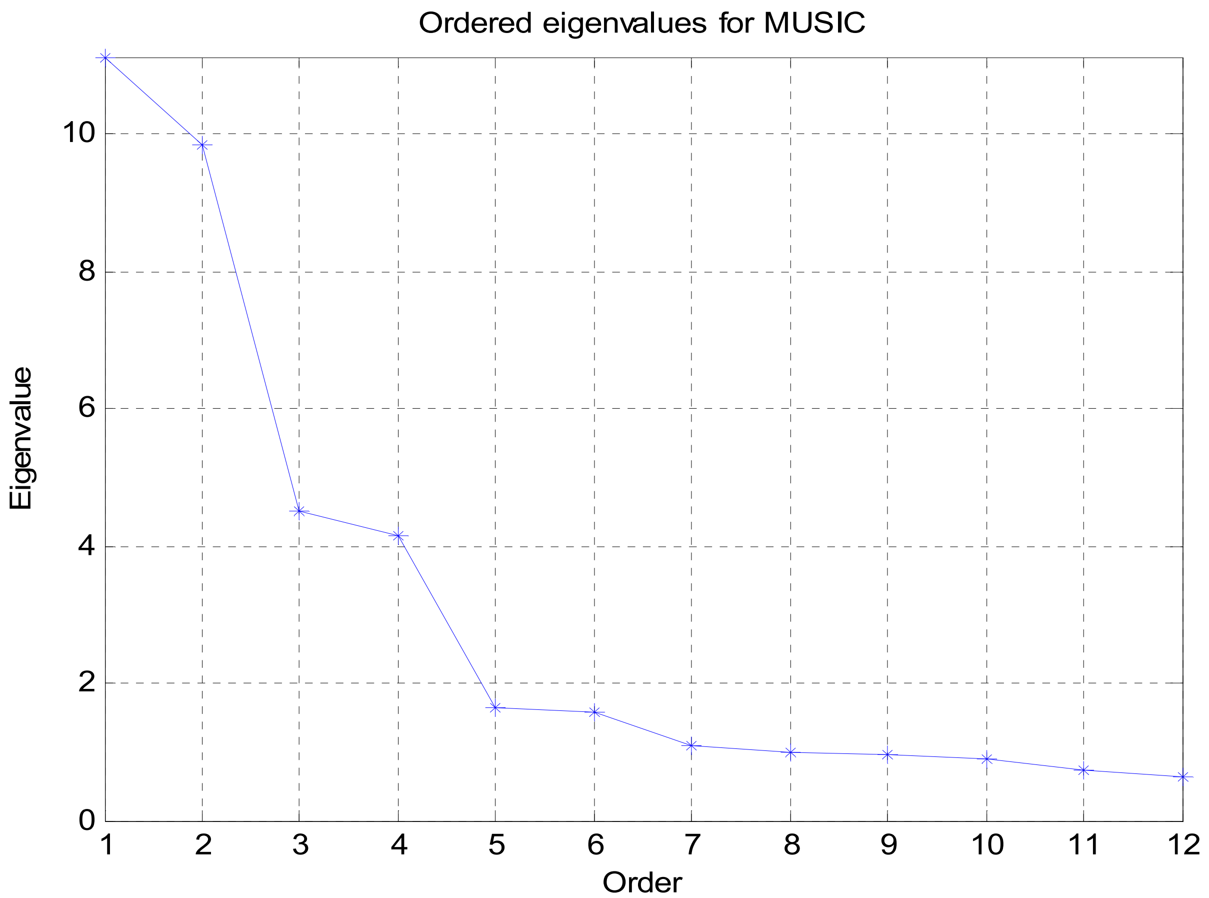
- A single source with low signal to noise ratio

- When sources are closely located the first eigenvalue increases and the second decreases

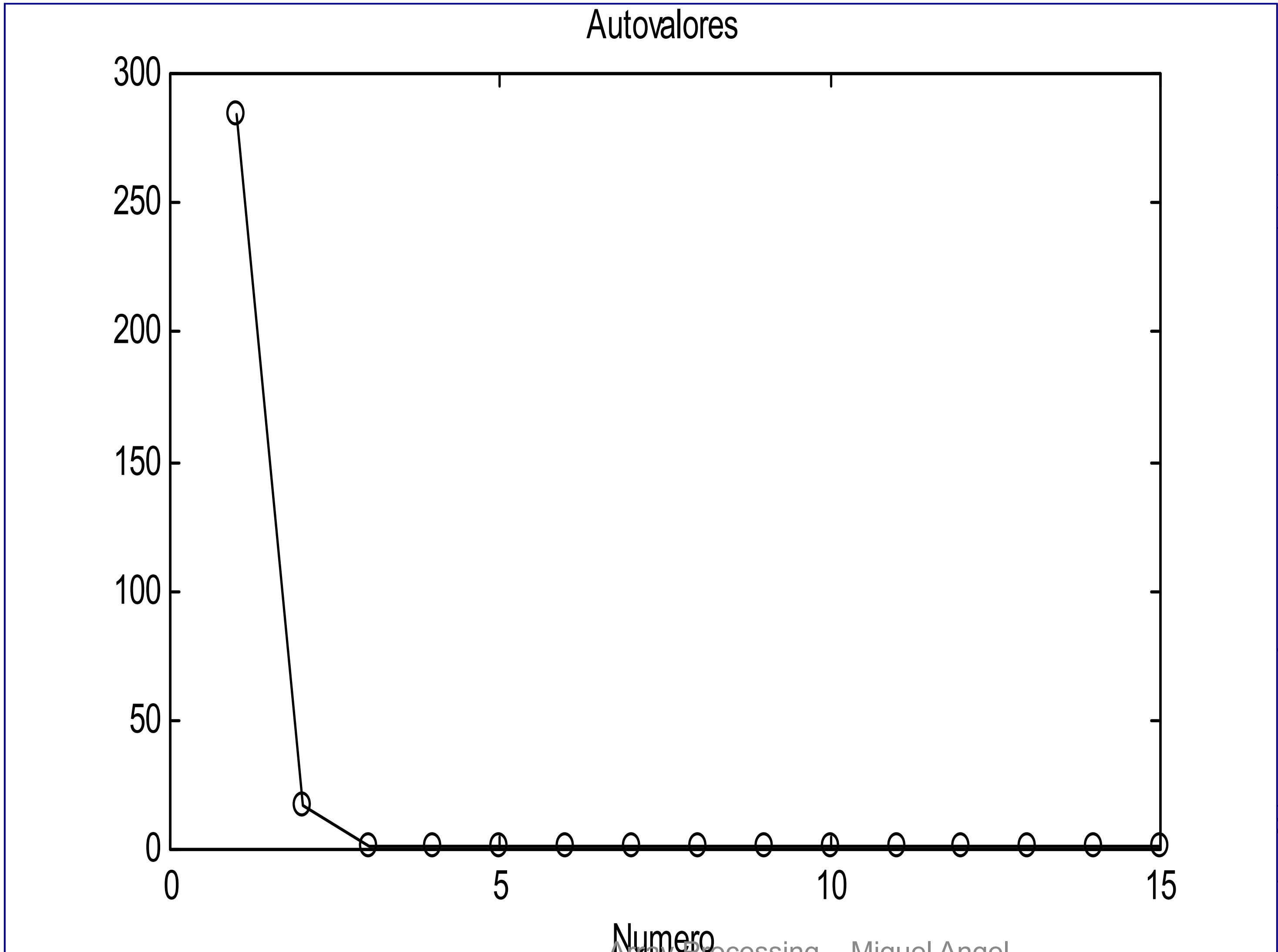
Example: NS=2 (Two actual sources)
Note that eigenvectors are beams (Eigenbeams).
The maximum takes maximum energy from all sources present.
The minimum does the same but with the constraint of
being orthogonal to the previous one.

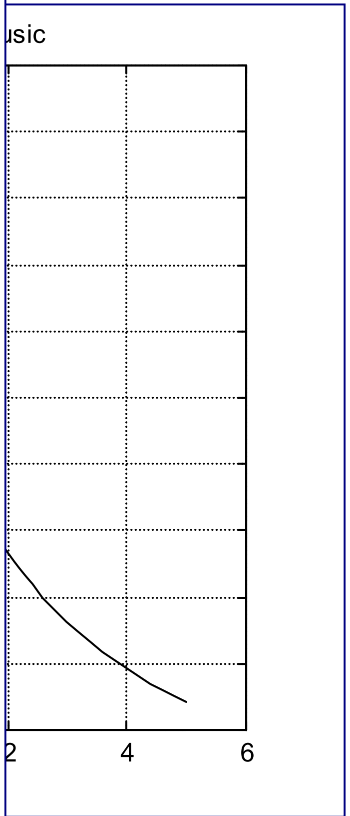


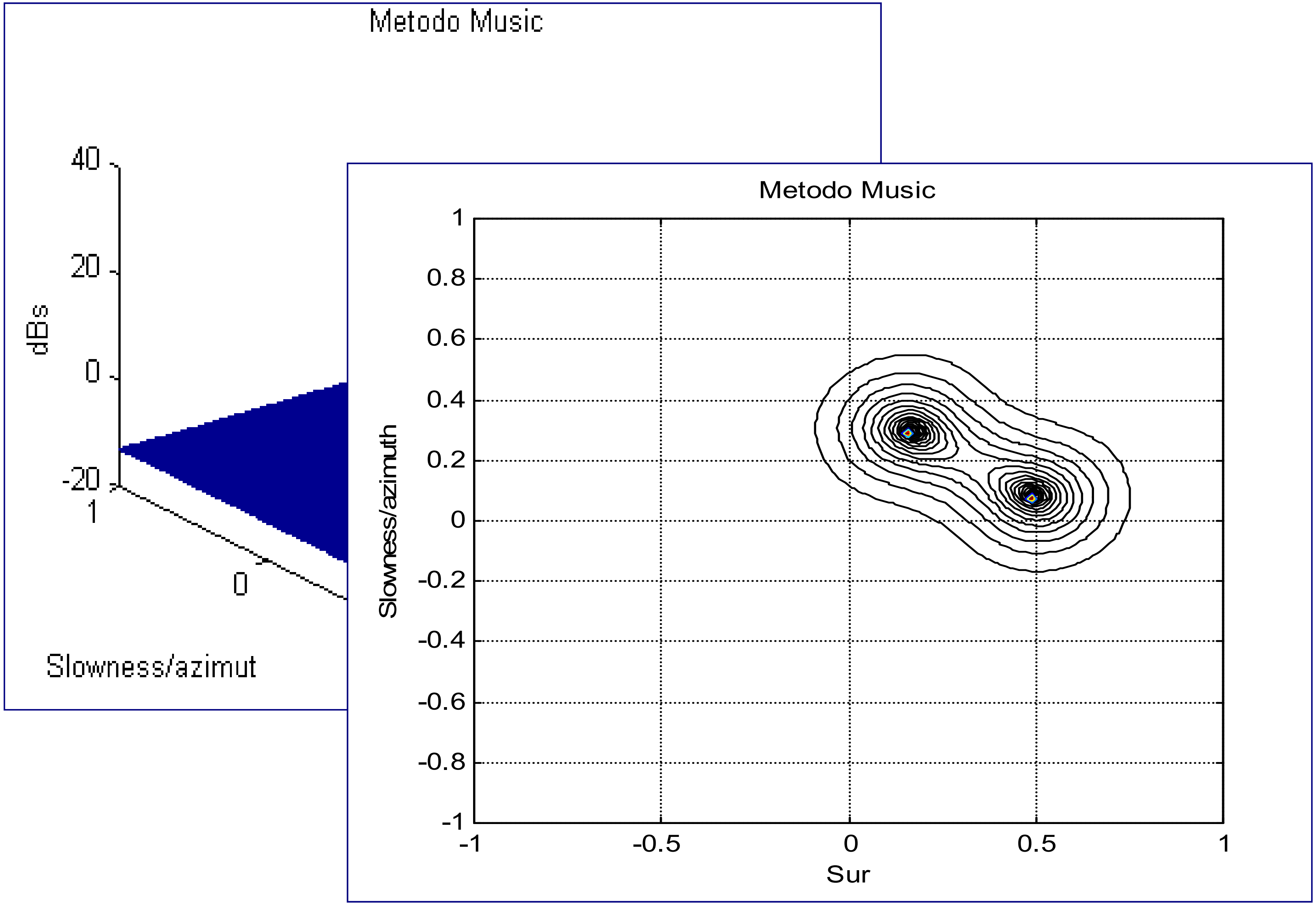
Difficult to decide the signal subspace dimenssion !!!



Music Performance






Metodo Music
40
20
dBs
0
-20
1
0
Slowness/azimut
Metodo Music
Slowness/azimuth
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
Sur

DOA Enhanced Methods from noise subspace versions.

Pisarenko predicted MUSIC by further generalizing MUSIC (Almost 50 years before MUSIC was reported) by the so-called potential density estimators.

First, note that any positive or negative power of a covariance matrix is reflected in its eigenvalues

$$\underline{\underline{R}}^{m} = \sum_{q=1}^{Q} \lambda_q^m . \underline{e}_q . \underline{e}_q^H$$

As a consequence taking the Capon's power density estimate,

$$P^{MLM}(\underline{S}) = \frac{1}{\underline{S}^H . \underline{\underline{R}}^{-1} . \underline{S}}$$

Can be written as: $P^{MLM}(\underline{S}) = \dfrac{1}{\sum_{q=1}^{Q} \lambda_q^{-1} . \left| \underline{S}^H . \underline{e}_q \right|^2}$

Knowing that noise eigenvalues are orthogonal to the steering vectors of the sources, artificial resolution can be added by just using only the noise subspace. This is refereed as the noise subspace version of Capon's method and was reported by Jhonson.

$$\Phi^{JH}(\underline{S}) = \frac{1}{\sum_{q=NS+1}^{Q} \lambda_q^{-1} \left| \underline{S}^H . \underline{e}_q \right|^2}$$

The noise subspace version of NMLM, also improving resolution is:

$$\Phi^{NMLM}(\underline{S}) = \frac{\sum_{q=1}^{Q} \lambda_q^{-1} . \left| \underline{S}^H . \underline{e}_q \right|^2}{\sum_{q=1}^{Q} \lambda_q^{-2} . \left| \underline{S}^H . \underline{e}_q \right|^2}$$

In fact, Pisarenko proved the following statement:

For any function f(x) such that the inverse exist and is continuous g(f(x))=x the family of estimates

$$\lim_{Q\to\infty}\left(\Phi^{ESTIMATE}(\underline{S}) = g\left(\underline{S}^H f\left(\underline{\underline{R}}\right)\underline{S}\right)\right) = \Phi^{ACTUAL}(\underline{S})$$

Converges to the actual density when the size of the aperture goes to infinity yet preserving average inter-element spacing. (Steering vectors become eigenvectors of signal and the eigenvalues the actual density)

In consequence, the following estimates converge asymptotically to the actual power distribution:

$$\Phi^{POT1}(\underline{S}) = \frac{\underline{S}^H.\underline{\underline{R}}^{-m+1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}^{-m}.\underline{S}} = \frac{\sum_{q=1}^{Q}\lambda_q^{-m+1}.\left|\underline{S}^H.\underline{e}_q\right|^2}{\sum_{q=1}^{Q}\lambda_q^{-m}.\left|\underline{S}^H.\underline{e}_q\right|^2}$$

$$\Phi^{POT2}(\underline{S}) = \frac{1}{\left[\underline{S}^H.\underline{\underline{R}}^{-m}.\underline{S}\right]^m} = \frac{\quad}{\left[\sum_{q=1}^{Q}\lambda_q^{-m}.\left|\underline{S}^H.\underline{e}_q\right|^2\right]^m}$$

$$\Phi^{LOG}(\underline{S}) = -Ln\left[\exp\left(\underline{S}^H.\underline{\underline{R}}^{-1}.\underline{S}\right)\right] = -Ln\left(\sum_{q=1}^{Q}\exp(-\lambda_q).\left|\underline{S}^H.\underline{e}_q\right|^2\right)$$

NOTE that MUSIC does not have this property since function f(x) is not invertible. So Music is a source detector but no an estimate of the power distribution versus angle

MUSIC for colored noise background

When the noise back-ground covariance is still full range and known a priori, still MUSIC can be used with the following modification:

Assuming that the model for the measured covariance is:

$$\underline{\underline{R}} = \underline{\underline{S}}.\underline{\underline{P}}.\underline{\underline{S}}^H + \underline{\underline{R}}_o =$$

Then---

$$= \sum_{s=1}^{NS} P(s)\underline{S}_s \underline{S}_s^H + \underline{\underline{R}}_0$$

$$\underline{\underline{R}}_0^{-1/2}.\underline{\underline{R}}.\underline{\underline{R}}_0^{-1/2} = \sum_{s=1}^{NS} P(s).\underline{b}_s \underline{b}_s^H + \underline{\underline{I}}$$

being

$$\underline{b}_s = \underline{\underline{R}}_0^{-1/2} \underline{S}_s$$

In consequence, the noise eigenvectors of matrix or generaliced noise eigenvectors of the matrix pencil R and R_0 are orthogonal to the new vector b_s

$$\underline{\underline{R}}_0^{-1/2}.\underline{\underline{R}}.\underline{\underline{R}}_0^{-1/2} \Rightarrow \underline{\underline{R}}\underline{e}_q = \lambda(q)\underline{\underline{R}}_0 \underline{e}_q$$

In summary the new estimate, will be constructed over the noise generalized eigenvectors of the pencil matrix R, R_0 and the sources will be located at the local maxima, among $\underline{S}$, of …..

$$\Phi(\underline{S}) = \frac{1}{\sum_{q=NS+1}^{Q} \left| \underline{b(\underline{S})}^H . \underline{u}_q \right|^2} = \frac{1}{\sum_{q=NS+1}^{Q} \left| \underline{S}^H . \underline{\underline{R}}_0^{-1/2} . \underline{u}_q \right|^2}$$

Note that the selection of the dimension stays on the generalized problem as well as the difficulties on its estimation from data covariance matrixes.

Scanning complexity on DOA estimation

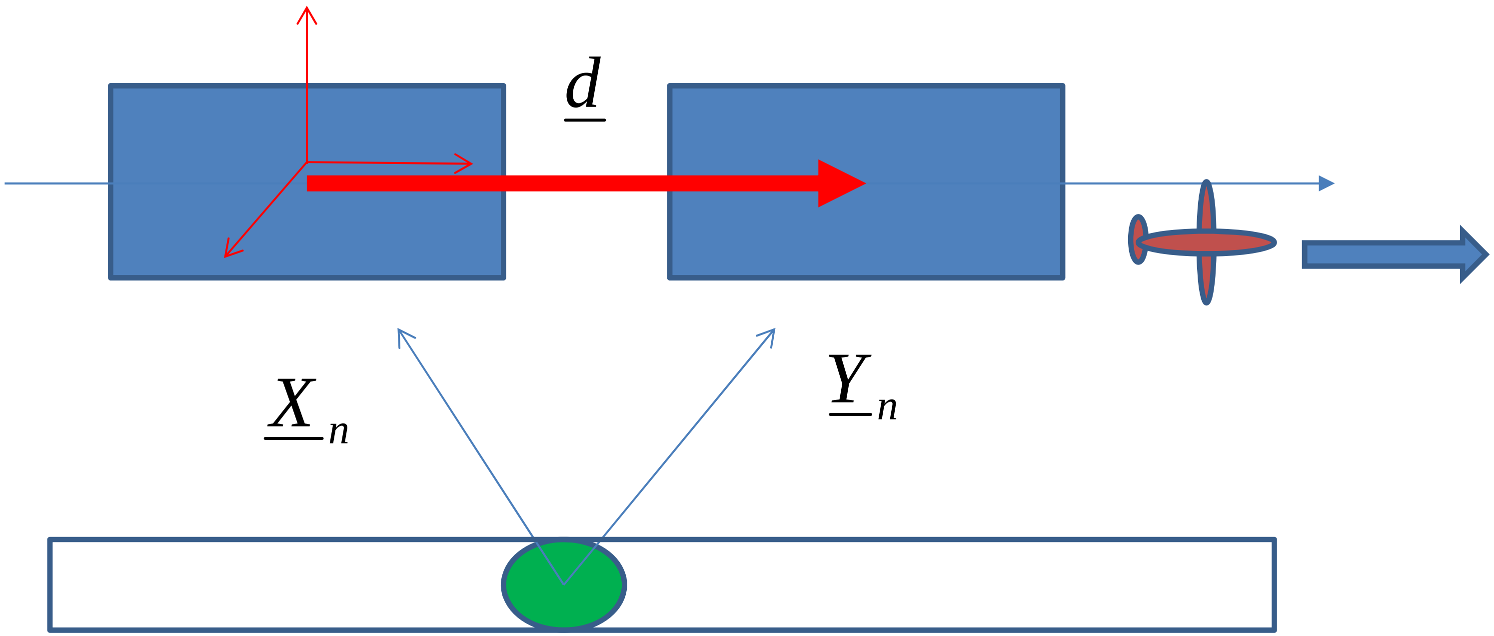
For resolution close to 0.1º, the scanning, trough vector S, in all the estimates reported, implies the computation of a QxQ quadratic form (360*10).(180*10)=6.480.000 times (!!!!!!).

This represents a hard limitation for implementing these procedures on limited resource array stations.

- Limited field of view
- Selective scanning
- ESPRIT using moving platform or twin apertures.

The basic idea is to perform spatial linear prediction and solving it as an exact problem using singular value decomposition SVD.

ESPRIT



After target illumination, two set of snapshots, X and Y are collected from two virtual apertures separated a vector d, motivated by the constant velocity movement of the platform.

$$\underline{Z}_n = \begin{bmatrix} \underline{X}_n \\ \underline{Y}_n \end{bmatrix}$$

Is the 2Q size global snapshot

Our target is to design a spatial linear predictor T such that the second snapshot is predicted from the first minimizing the prediction error

$$\left[-\underline{\underline{T}} \quad \underline{\underline{I}}\right]\underline{Z}_n = \underline{\varepsilon}_n \Rightarrow \left[-\underline{\underline{T}} \quad \underline{\underline{I}}\right]\begin{bmatrix} \underline{X}_n \\ \underline{Y}_n \end{bmatrix} = \underline{\varepsilon}_n \Rightarrow \boxed{\underline{Y}_n = \underline{\underline{T}}\,\underline{X}_n + \underline{\varepsilon}_n}$$

Regardless the procedure is coined as ESPRIT, the idea was original from J. Munier (Grenoble) and it was named as “Propagateur”. The Propagateur was used, not just for DOA estimation but also for callibration of towed sonar arrays.

Before solving the LP problem we will reduce noise using the svd of the covariance of the global snapshot.

Using the svd of covariance R_z

$$\underline{\underline{R}}_z = \frac{1}{N}\sum_{q=1}^{N}\underline{Z}_n\underline{Z}_n^H = \boxed{\underline{\underline{E}}_s}\,\underline{\underline{\Lambda}}_s\,\underline{\underline{E}}_s + \underline{\underline{E}}_n\,\underline{\underline{\Lambda}}_n\,\underline{\underline{E}}_n$$

Signal sub-space of dimension NS (equal to the actual number of sources

It is clear that the signal subspace eigenvector can be decomposed on the two apertures contributions

$$\underline{\underline{E}}_s = \begin{bmatrix} \underline{\underline{E}}_x \\ \underline{\underline{E}}_y \end{bmatrix}$$

In addition, we know that the signal eigenvectors of each aperture are related to the corresponding DOAs by a rotation matrix.

$$\underline{\underline{E}}_x = \underline{\underline{S}}\,\underline{\underline{G}}$$

$$\underline{\underline{E}}_y = \underline{\underline{S}}\,\underline{\underline{\Phi}}\,\underline{\underline{G}}$$

These two contributions are QxNS size

where

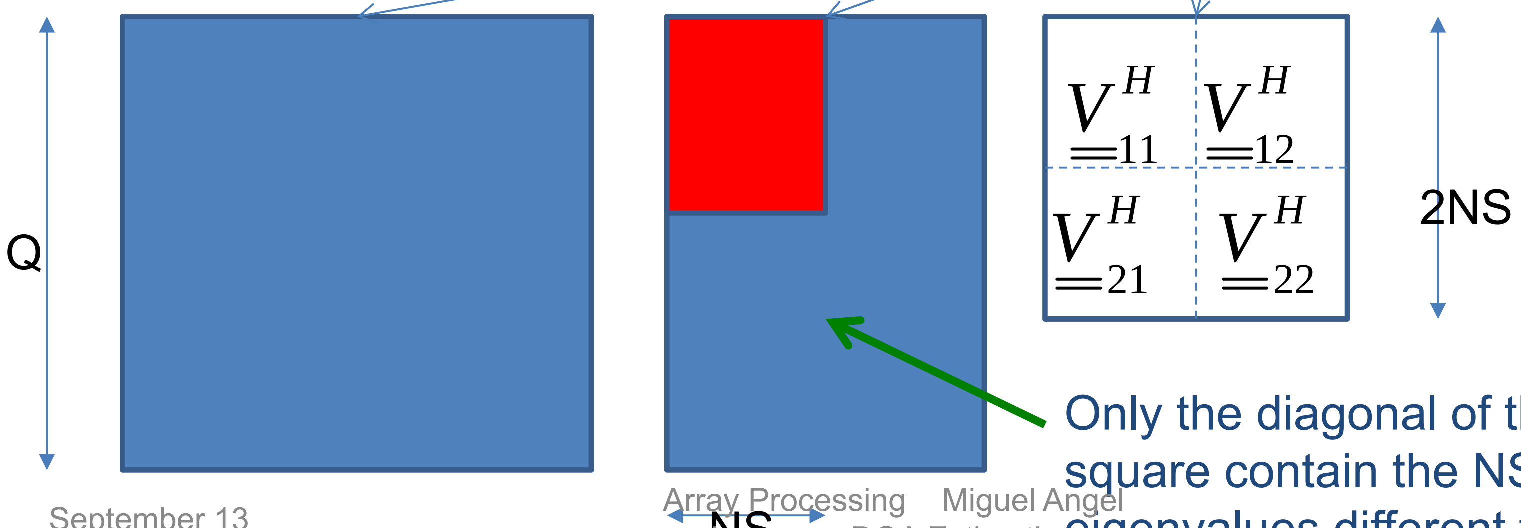
$$\underline{\underline{\Phi}} = diag\left(s = 1, NS \quad \exp\left(j\frac{2\pi f_c}{c}\underline{k}_{s(unitary)}^H\underline{d} \right)\right)$$

THE PROBLEM NOW IS TO DESIGN THE LINEAR PREDICTION BETWEEN THE TWO SUBSPACES. This can be formulated as finding F_1 and F_2 in this problem:

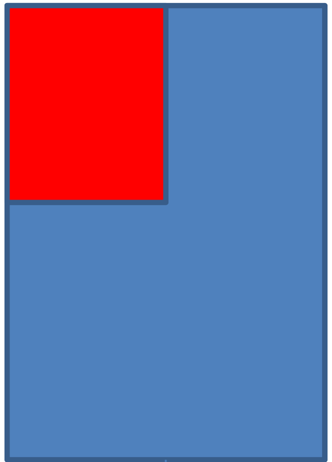
$$\begin{bmatrix} \underline{\underline{E}}_x & \underline{\underline{E}}_y \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{F}}_1 \\ \underline{\underline{F}}_2 \end{bmatrix} = \underline{\underline{\varepsilon}}$$

To solve the problem let us take a look of svd of the composition of the two subspaces

$$\begin{bmatrix} \underline{\underline{E}}_x & \underline{\underline{E}}_y \end{bmatrix} = \underline{\underline{U}}.\underline{\underline{D}}.\underline{\underline{V}}^H$$



Only the diagonal of the red square contain the NS eigenvalues different from zero



$$\begin{bmatrix} \underline{\underline{V}}_{11}^{H} & \underline{\underline{V}}_{12}^{H} \\ \underline{\underline{V}}_{21}^{H} & \underline{\underline{V}}_{22}^{H} \end{bmatrix} \quad \begin{bmatrix} \underline{\underline{V}}_{21} \\ \underline{\underline{V}}_{22} \end{bmatrix}$$

Multiplying by this matrix….

$$\begin{bmatrix} \underline{\underline{0}} \\ \underline{\underline{I}} \end{bmatrix}$$

This product produces zero or minimum residual. In consequence:

$$\begin{bmatrix} \underline{\underline{E}}_x & \underline{\underline{E}}_y \end{bmatrix} \begin{bmatrix} \underline{\underline{V}}_{12} \\ \underline{\underline{V}}_{22} \end{bmatrix} = \underline{\underline{0}} \quad \Rightarrow$$

$$-\underline{\underline{E}}_x \underline{\underline{V}}_{12} = \underline{\underline{E}}_y \underline{\underline{V}}_{22} \quad or$$

$$\underline{\underline{E}}_x \left(-\underline{\underline{V}}_{12} \underline{\underline{V}}_{22}^{-1} \right) = \underline{\underline{E}}_y$$

$$\underline{\underline{E}}_x \underline{\underline{T}} = \underline{\underline{E}}_y$$

$$\underline{\underline{E}}_x \underline{\underline{T}} = \underline{\underline{E}}_y$$

Now, having T after 2 svd, we resort the relationship between eigenvectors and steering vectors

$$\underline{\underline{E}}_x = \underline{\underline{S}}\,\underline{\underline{G}}$$

$$\underline{\underline{E}}_y = \underline{\underline{S}}\,\underline{\underline{\Phi}}\,\underline{\underline{G}}$$

…. And we obtain

$$\underline{\underline{S}}\,\underline{\underline{G}}\,\underline{\underline{T}} = \underline{\underline{S}}\,\underline{\underline{\Phi}}\,\underline{\underline{G}} \quad \Longrightarrow \quad \underline{\underline{G}}\,\underline{\underline{T}} = \underline{\underline{\Phi}}\,\underline{\underline{G}}$$

And, finally….

$$\boxed{\underline{\underline{T}} = \underline{\underline{G}}^{-1}\,\underline{\underline{\Phi}}\,\underline{\underline{G}}}$$

After 3 svd we get the NS DOAs from the eigenvalues of matrix T. This completes the ESPRIT procedure

MAXIMUM LIKELIHOOD

Point Source DOA estimation

First at all, let us solve the problem of a single source in white noise.

$$\Pr\left(\underline{X}_n / a_n, \underline{S}, \underline{\underline{R}}_0\right) \propto \frac{1}{\det\left(\underline{\underline{R}}_0\right)} \exp\left[-\left(\underline{X}_n - a_n \underline{S}\right)^H \underline{\underline{R}}_0^{-1} \left(\underline{X}_n - a_n \underline{S}\right)\right]$$

The ML estimate of the complex envelope is:

$$\hat{a}_n = \frac{\underline{S}^H \underline{\underline{R}}_0^{-1} \underline{X}_n}{\underline{S}^H \underline{\underline{R}}_0^{-1} \underline{S}} = \frac{\underline{S}^H \underline{X}_n}{Q}$$

Note that to recover the envelope estimate we need the location or steering of the source. Now using the previous estimate in the likelihood we have:

$$\Pr\left(\underline{X}_n / \underline{S}, \sigma^2\right) \propto \frac{1}{\left(\sigma^2\right)^Q} \exp\left[-\left(\underline{X}_n - \underline{S} \frac{\underline{S}^H \underline{X}_n}{Q}\right)^H \frac{1}{\sigma^2} \left(\underline{X}_n - \underline{S} \frac{\underline{S}^H \underline{X}_n}{Q}\right)\right]$$

After some manipulations on the exponent….

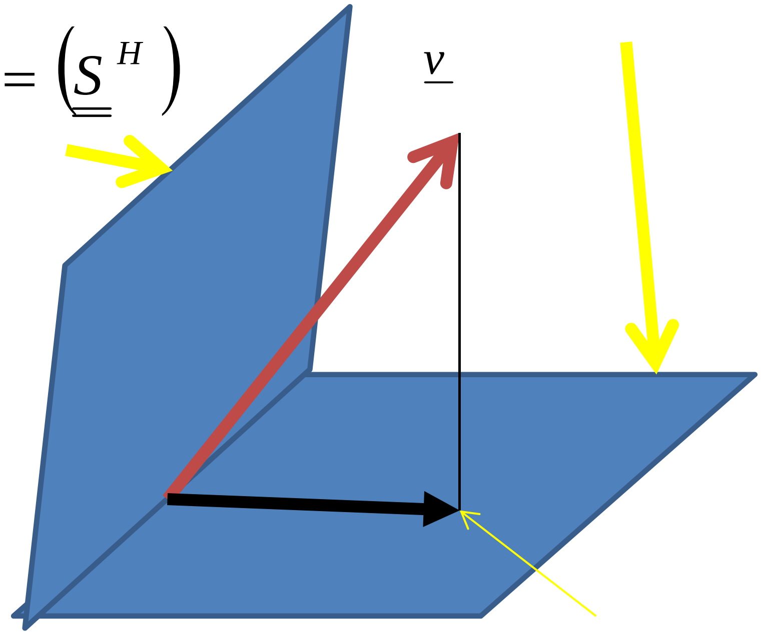
$$\left(\underline{X}_t - \underline{S}\frac{\underline{S}^H \underline{X}_t}{Q}\right)^H \frac{1}{\sigma^2}\left(\underline{X}_t - \underline{S}\frac{\underline{S}^H \underline{X}_t}{Q}\right) =$$

$$= \underline{X}_t^H \left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q}\right)\left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q}\right)\underline{X}_t \frac{1}{\sigma^2} = \underline{X}_t^H \left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q}\right)\underline{X}_t \frac{1}{\sigma^2}$$

P is a projection operator.

$$\underline{\underline{P}} = \left(\underline{\underline{I}} - \underline{\underline{S}}\left(\underline{\underline{S}}^H \underline{\underline{S}}\right)^{-1} \underline{\underline{S}}^H\right)$$

$$\underline{\underline{P}}_{=} = \left(\underline{\underline{S}}^H\right)$$



$$\underline{\underline{P}}_{\perp}\underline{v}$$

This operator provides the projection on the orthogonal subspace to the space defined by matrix S

This P generates a projection of the received snapshot on the orthogonal subspace of S, i.e. removes the content from direction S from the data snapshot.

$$\left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q}\right)\underline{X}_t$$

Since the trace of a scalar is the same scalar and the trace is circular then….

$$\frac{1}{\sigma^2} tr\left(\underline{X}_t^H \left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q} \right) \underline{X}_t \right) = \frac{1}{\sigma^2} tr\left(\left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q} \right) \underline{X}_t \underline{X}_t^H \right)$$

$$tr\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = tr\left(\underline{\underline{B}}\,\underline{\underline{A}}\right)$$

$$tr\left(\underline{a}^H \underline{b}\right) = scalar \;\; = \underline{a}^H \underline{b} = tr\left(\underline{b}\,\underline{a}^H\right)$$

To continue the estimation we need additional snapshots N. These snapshots are independent (since the noise is white on the time domain), and, in consequence the probability will be the product of the N single probabilities.

$$\prod_1^M \Pr\left(\underline{X}_t / \underline{S}, \underline{\underline{R}}_0\right) \propto \left(\sigma^2\right)^{-QM} \exp\left[-\left(\underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q} \right) \sum_1^M \underline{X}_t \underline{X}_t^H \right]$$

$$Ln\left(\prod_1^N \Pr\left(\underline{X}_n / \underline{S}, \underline{\underline{R}}_0\right) \right) \propto -QNLn\left(\sigma^2\right) - \frac{1}{\sigma^2} tr\left(N\,\underline{\underline{P}}\,\underline{\underline{R}}\right)$$

$$being \qquad \underline{\underline{P}} = \underline{\underline{I}} - \frac{\underline{S}\,\underline{S}^H}{Q} \quad y \quad \underline{\underline{R}} = \frac{1}{M} \sum_1^N \underline{X}_n \underline{X}_n^H$$

Now we can go for the estimate of the noise level just taking derivative of log-likelihood function

$$-QM\frac{1}{\sigma^2}+\frac{1}{\left(\sigma^2\right)^2}tr\left(M\,\underline{\underline{P}}\,\underline{\underline{R}}\right)=0$$

The resulting estimate is…………

$$\boxed{\sigma^2_{ML}=tr\left(\underline{\underline{P}}\,\underline{\underline{R}}\right)/Q}$$

Using the noise estimate at the log-likelihood

$$Ln\left(\prod_1^M \Pr\left(\underline{X}_n/\underline{S}\right)\right)\propto -QMLn\left(\sigma^2_{ML}\right)+Q$$

Since the maximum among S, this implies that the proper S has to minimice the noise power estimate.

$$MIN_{\underline{S}}\,\sigma^2_{ML}=MIN_{\underline{S}}\,tr\left(\underline{\underline{P}}\,\underline{\underline{R}}\right)/Q=MIN_{\underline{S}}\frac{1}{Q}tr\left(\underline{\underline{R}}-\frac{\underline{S}\,\underline{S}^H\underline{\underline{R}}}{Q}\right)=$$

$$=MAX_{\underline{S}}\,tr\left(\frac{\underline{S}\,\underline{S}^H\underline{\underline{R}}}{Q}\right)=MAX_{\underline{S}}\frac{\underline{S}^H\underline{\underline{R}}\,\underline{S}}{Q}=MAX_{\underline{S}}\frac{1}{QM}\sum_1^M\left|\underline{S}^H\underline{X}_n\right|^2$$

When the number of sources is greater than one complexity grows making almost unpractical the ML estimate

$$\underline{X}_n = \underline{\underline{S}}.\underline{a}_n + \underline{w}_n$$

The ML estimate of the sources' waveforms is:

$$\underline{\hat{a}}_n = \left(\underline{\underline{S}}^H.\underline{\underline{S}}\right)^{-1}.\underline{\underline{S}}^H.\underline{X}_n$$

And the new likelihood…….

$$\Pr(\underline{\underline{X}}/\underline{\underline{S}},\underline{\underline{Q}}) = \frac{1}{\left(\pi.\sigma^{2Q}\right)^N}.\exp\left\{-\sum_{n=0}^{N-1}\underline{X}_n^H.\underline{\underline{P}}_\oplus.\underline{\underline{P}}_\oplus.\underline{X}_n/\sigma^2\right\}$$

where

$$\underline{\underline{P}}_\oplus = \underline{\underline{I}} - \underline{\underline{S}}.\left(\underline{\underline{S}}^H.\underline{\underline{S}}\right)^{-1}.\underline{\underline{S}}^H$$

And the log-likelihood

$$L(\underline{\underline{S}},\sigma) = k_0 - N.Q.Ln\left(\sigma^2\right) - \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left|.\underline{\underline{P}}_\oplus.\underline{X}_n\right|^2$$

The ML estimate of the noise level is….

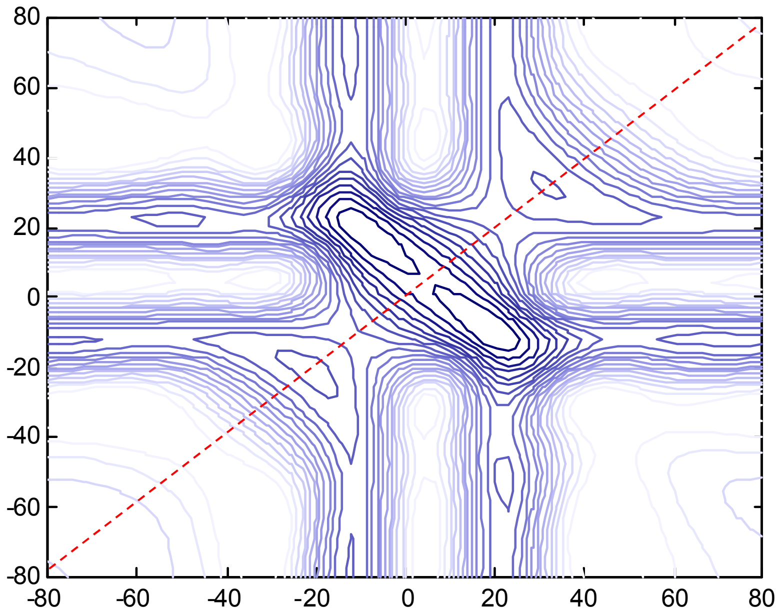
$$\hat{\sigma}^2 = \frac{1}{N.Q}.\sum_{n=0}^{N-1}\left|\underline{\underline{P}}_\oplus.\underline{X}_n\right|^2$$

At the end the function to be minimized is quite similar to the case of a single source. Nevertheless it implies a double search over the to angles simultaneously. For NS sources this implies a simultaneous search over NS angles or 2NS angles for planar apertures

$$L(\underline{\underline{S}}) = \sum_{n=0}^{N-1} \left| \underline{\underline{P}}_{\oplus} \cdot \underline{X}_n \right|^2$$

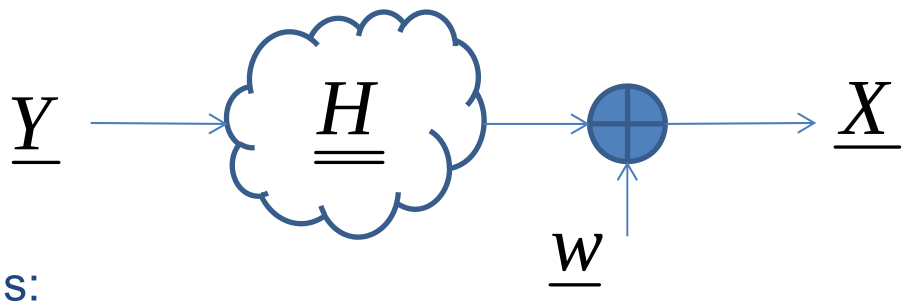
$$L(\underline{\underline{S}}) = Traza\left[\underline{\underline{P}}_{\oplus} \cdot \underline{\underline{\hat{R}}} \right]$$

In addition, L(S) function is smooth which almost precludes any accelerated search of local maxima. An example can be viewed for two sources in an ULA array located at -21 and 10 degrees.



THE EM Algorithm

PRELIMINAR: The MAP Estimate, the problem is given the statistics (mean and covariance) of vectors Y and X, we like to have a MAP estimate of Y given X.

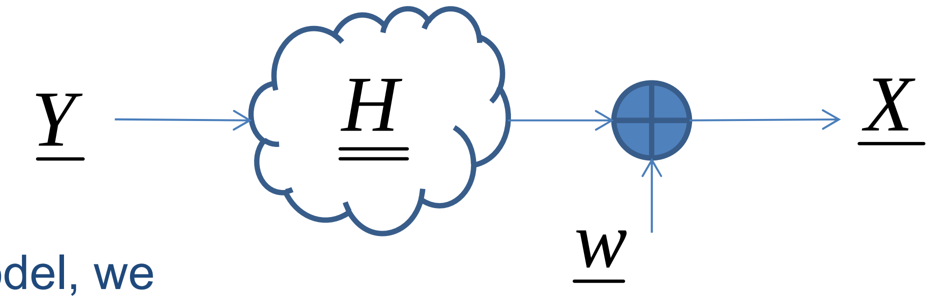


The estimate is:

$$\hat{\underline{Y}} = \underline{m}_y + \underline{\underline{C}}_{yx} . \underline{\underline{C}}_{xx}^{-1} \left(\underline{X} - \underline{m}_x \right)$$

$$being : \underline{m}_y = E\left(\underline{Y}\right) \quad and \ \underline{\underline{C}}_{yx} = E\left(\left(\underline{Y} - \underline{m}_y \right) \left(\underline{X} - \underline{m}_x \right)^H \right)$$

$$\text{with covariance} \quad \underline{\underline{\mathrm{C}}}_{\hat{y}\hat{y}} = \underline{\underline{C}}_{yy} - \underline{\underline{C}}_{yx} \underline{\underline{C}}_{xx}^{-1} \underline{\underline{C}}_{xy}$$



Using the model, we have……..

$$\underline{m}_x = \underline{\underline{H}}.\underline{m}_y \qquad \underline{\underline{C}}_{yx} = \underline{\underline{C}}_{yy}\underline{\underline{H}}^H \qquad \underline{\underline{C}}_{xx} = \underline{\underline{H}}.\underline{\underline{C}}_{yy}\underline{\underline{H}}^H + \sigma_x^2 \underline{\underline{I}}$$

$$\hat{\underline{Y}} = \underline{m}_y + \underline{\underline{C}}_{yy}\underline{\underline{H}}^H \underline{\underline{C}}_{xx}^{-1}\left(\underline{X} - \underline{\underline{H}}\,\underline{m}_y\right)$$

When

$$\underline{Y} = \underline{m}_y + \underline{w}_y \Rightarrow \underline{\underline{C}}_{yy} = \sigma_y^2 \underline{\underline{I}}_{Q.NS} \qquad \underline{\underline{C}}_{xx} = \left(\frac{\sigma_x^2}{NS}\right)\underline{\underline{H}}\,\underline{\underline{H}}^H + \sigma^2 \underline{\underline{I}}_Q$$

$$\boxed{\hat{\underline{Y}} = \underline{m}_y + \underline{\underline{H}}^H \underline{\underline{C}}_{xx}^{-1}\left(\underline{X} - \underline{\underline{H}}\,\underline{m}_y\right)}$$

The Estimate and Maximice

Observed Data → Snapshot → $\underline{X}_n$

UN-COMPLETE DATA

Desired Data → Single source snapshot → $\underline{Y}_n$

COMPLETE DATA

Modeling un-complete and complete data ....

$$\underline{X}_n = \sum_{s=1}^{NS} a_s(n)\underline{S}_n + \underline{w}_n$$

$$\underline{m}_x = \sum_{s=1}^{NS} a_s(n)\underline{S}_n$$

$$\underline{\underline{C}}_{xx} = \sigma^2 \underline{\underline{I}}$$

$$\underline{Y}_n = \begin{bmatrix} \underline{Y}_{1n} \\ .. \\ \underline{Y}_{sn} \\ .. \\ \underline{Y}_{NSn} \end{bmatrix} = \begin{bmatrix} a_1(n)\underline{S}_1 + \underline{w}_{1n} \\ .. \\ a_s(n)\underline{S}_s + \underline{w}_{sn} \\ .. \\ a_s(n)\underline{S}_{NS} + \underline{w}_{NSn} \end{bmatrix}$$

$$\underline{m}_y = \begin{bmatrix} a_1(n)\underline{S}_1 \\ .. \\ a_s(n)\underline{S}_s \end{bmatrix}$$

$$\underline{X}_n = \underline{\underline{H}}\,\underline{Y}_n = \begin{bmatrix} \underline{\underline{I}}_Q & .... & \underline{\underline{I}}_Q \end{bmatrix} \underline{Y}_n$$

$$\underline{\underline{C}}_{yy} = \left(\frac{\quad}{NS}\right) \underline{\underline{I}}_{Q.NS}$$

Going back to the ML estimate

Let us group on vector Ө all the parameters we like to estimate, i.e. the source waveforms and the corresponding steering vectors, the likelihood we would like to maximize is:

$$\Pr\left(\underline{X}_n / \underline{\vartheta} \right) \Rightarrow$$

In order to connect this with the complete data we use Bayes theorem.

$$\Pr\left(\underline{X}_n / \underline{\vartheta} \right) = \frac{\Pr\left(\underline{Y}_n / \underline{\vartheta} \right)}{\Pr\left(\underline{Y}_n / \underline{X}_n, \underline{\vartheta} \right)} \Pr\left(\underline{X}_n / \underline{Y}_n, \underline{\vartheta} \right)$$

Since, given the complete data the un-complete data can be found directly, without the need of the sources' parameters, in consequence this term do not impacts on the maximization of the likelihood

In terms of the log-likelihood the function to be maximized is:

$$\Im\left(\underline{X}_n / \underline{\vartheta}\right) \propto \Im\left(\underline{Y}_n / \underline{\vartheta}\right) - \Im\left(\underline{Y}_n / \underline{X}_n, \underline{\vartheta}\right)$$

Let us assume we have a prior of the parameters $\Theta^{(m)}$ at step m this allows us to have the mean of the un-complete data. Is to The objective is to obtain a better estimate $\Theta^{(m+1)}$

$$\Im\left(\underline{Y}_n / \underline{\theta}^{(m)}, \underline{\theta}^{(m+1)}\right) - \Im\left(\underline{Y}_n / \underline{X}_n, \underline{\theta}^{(m)}, \underline{\theta}^{(m+1)}\right) =$$

$$= U\left(\underline{\theta}^{(m)}, \underline{\theta}^{(m+1)}\right) - V\left(\underline{\theta}^{(m)}, \underline{\theta}^{(m+1)}\right)$$

$$U\left(\underline{\theta}^{(m)},\underline{\theta}^{(m+1)}\right)-V\left(\underline{\theta}^{(m)},\underline{\theta}^{(m+1)}\right)$$

Since U entails the NS single source likelihood problems, solving these NS problems--→ MAXIMIZE we will find an improvement on U. In summary:

$$U\left(\underline{\theta}^{(m)},\boxed{\underline{\theta}^{(m+1)}}\right)\geq U\left(\underline{\theta}^{(m)},\underline{\theta}^{(m)}\right)$$

This is set in order to maximize U

Since V entails the MAP estimate of Y given X and the prior of the parameters, whenever we use a different parameter vector from the prior the function will decrease → ESTIMATE. In summary:

$$V\left(\underline{\theta}^{(m)},\underline{\theta}^{(m+1)}\right)\leq V\left(\underline{\theta}^{(m)},\underline{\theta}^{(m)}\right)$$

ITERATING OVER ESTIMATE and MAXIMIZE STEPS WE WILL IMPROVE THE MULTIPLE SOURCES LIKELIHOOD solving the problem.

The MAXIMIZE step

This step reduces to solve the ML problem for a single source in white noise.

- NOTE THAT the sources' waveforms estimation implies to perform EM steps at the single snapshot level. At the same time, noise in the complex envelope may require several iterations in order to reduce the effects of the estimation noise on the following estimate step.

- The need to estimate the source waveform is the major drawback of the EM procedure.

- Convergence is fast for reduces number of sources

- The procedure is robust to coherent sources
- The weight 1/NS comes from assuming the noise on ion of the complete data a fraction of the global noise. When it is assumed that the noise on each complete data is equal to the global noise, the coefficient reduces to 1.

The ESTIMATE step

Given the source parameters from the prior, we compose the mean of the sources vector…..

$$\underline{m}_y^{(m)} = \begin{bmatrix} a_1^{(m)}(n)\underline{S}_1^{(m)} \\ .. \\ a_s^{(m)}(n)\underline{S}_s^{(m)} \\ .. \\ a_{NS}^{(m)}(n)\underline{S}_{NS}^{(m)} \end{bmatrix}$$

$$\underline{\underline{C}}_{yy} = \left(\frac{\sigma^2}{NS}\right)\underline{\underline{I}}_{Q.NS}$$

Then, we obtain the complete data as:

$$\boxed{\begin{aligned} \underline{\hat{Y}}_n^{(m+1)} &= \underline{m}_y^{(m)} + \frac{1}{NS}\underline{\underline{H}}^H\left(\underline{X}_n - \underline{\underline{H}}\,\underline{m}_y\right) = \\ &= \underline{m}_y^{(m)} + \left(\frac{1}{NS}\right)\begin{bmatrix} \underline{\underline{I}}_Q \\ .. \\ \underline{\underline{I}}_Q \end{bmatrix}\left(\underline{X}_n - \underline{\underline{H}}\,\underline{m}_y\right) \end{aligned}}$$

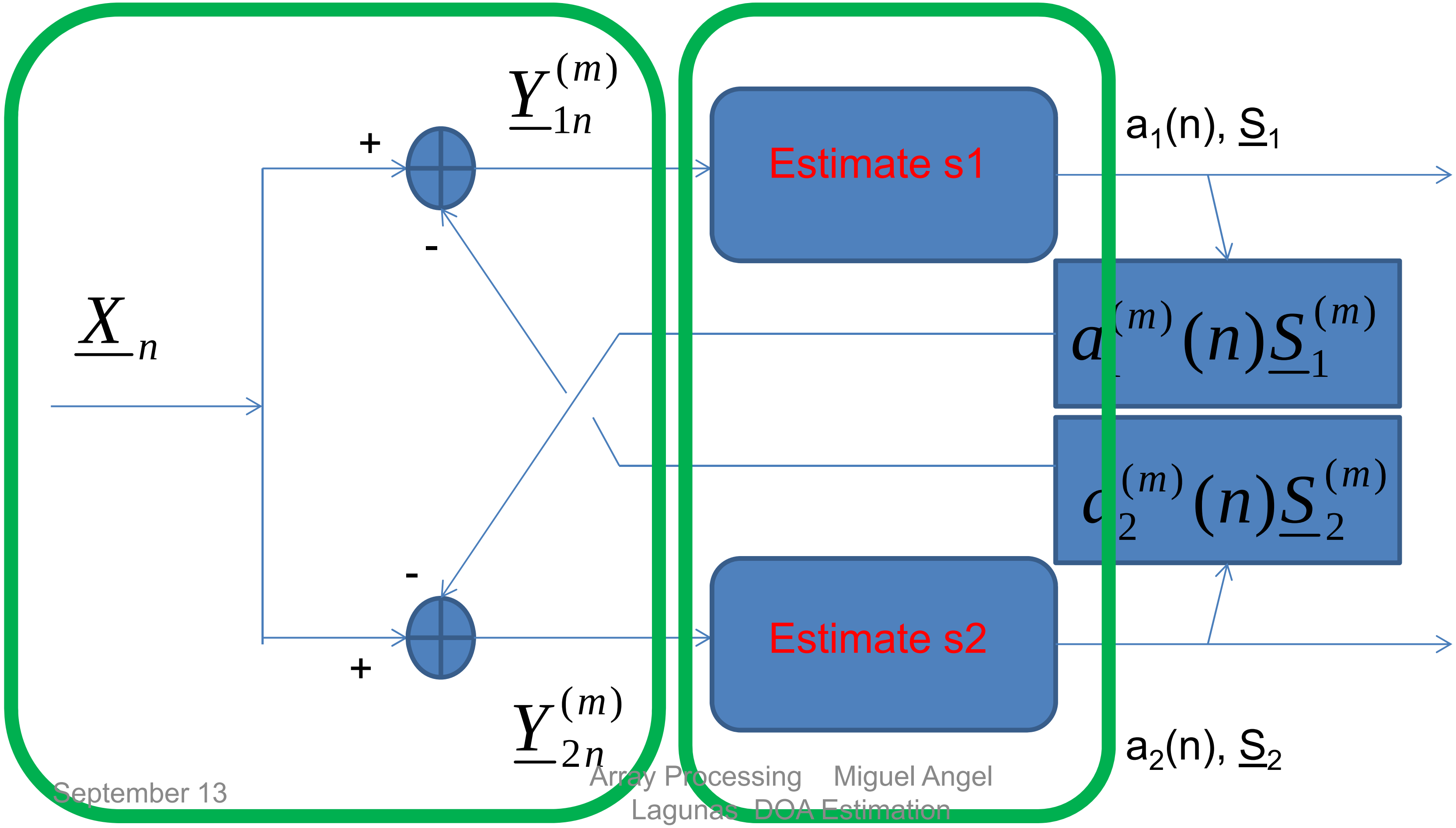
$$\underline{\hat{Y}}_n^{(m+1)} = \underline{m}_y^{(m)} + \begin{bmatrix} \underline{\underline{I}}_Q \\ \cdot\cdot \\ \underline{\underline{I}}_Q \end{bmatrix} \left(\underline{X}_n - \underline{\underline{H}}\,\underline{m}_y \right)$$

$$\underline{\hat{Y}}_{sn}^{(m+1)} = \underline{X}_n - \sum_{\substack{q=1 \\ q \neq s}}^{NS} \hat{a}_q^{(m)} \underline{S}_q^{(m)}$$

The EM becomes very intuitive: To generate the data for a single source the rest of the sources contribution is subtracted (is removed) from the received snapshot

EM Architecture

Without loss of generality we can draw the processing architecture of the EM algorithm for the case of two sources.



The Alternate Projection algorithm

Some, less formal, versions of the EM has been used in practice in field like radar (CLEAN) as well as sub-optimal procedures which overpass the source waveform estimation like the AP. AP is a concept used on linear programing methods an it is well known in mathematics.

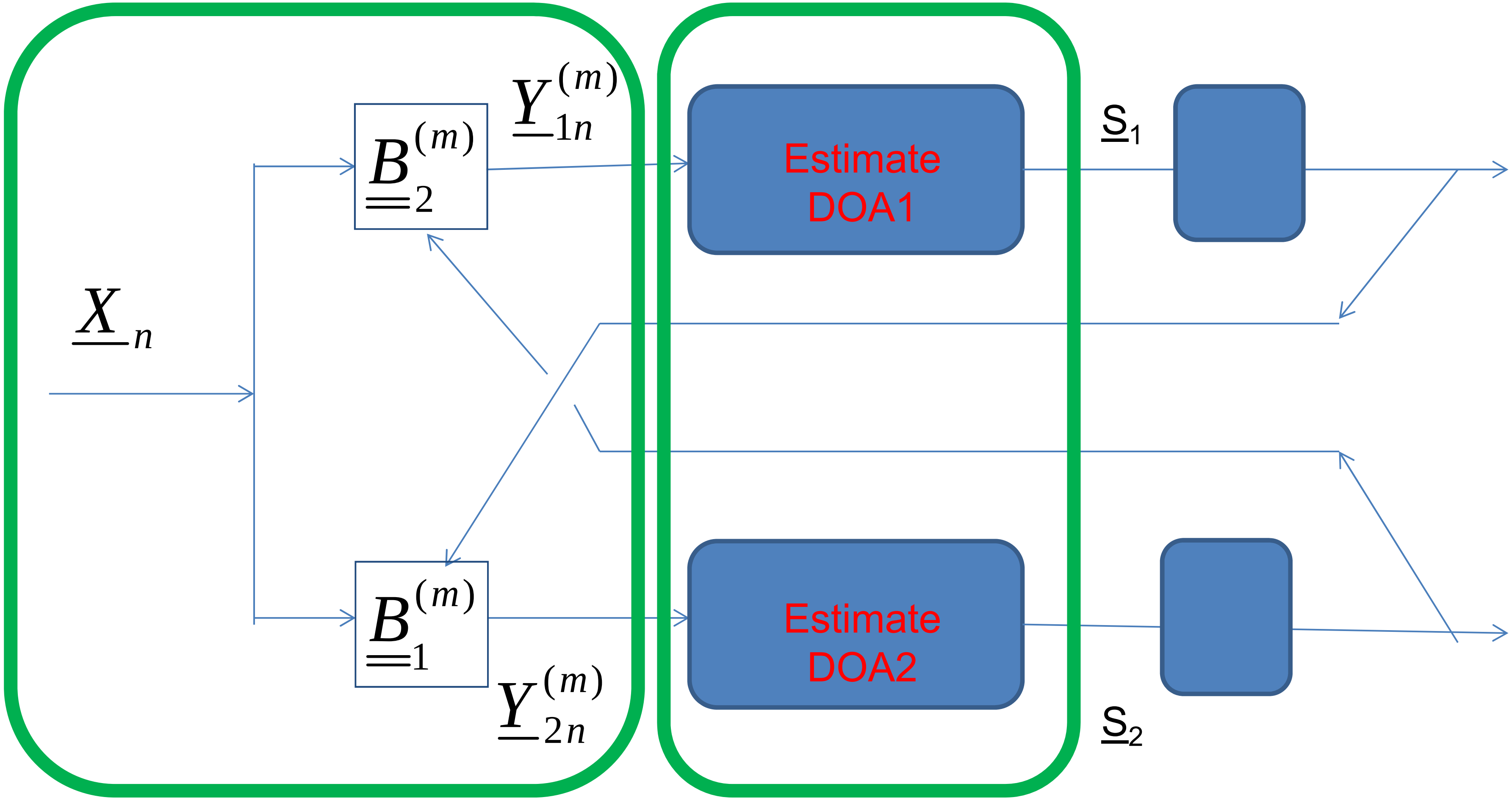
The basic idea of the EM remains in the sense that passing from the original snapshot to the single source problem we need to remove the other source effects. Nevertheless, in order to remove the other sources we use blocking the spatial directions instead of subtracting waveforms.

As an example, this matrix may be used to source s from a snapshot

$$\begin{bmatrix} 1 & -\exp(j(u_{s2}-u_{s1}) & 0 & . & 0 \\ 0 & 1 & -\exp(j(u_{s3}-u_{s2}) & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & \exp(j(u_{sQ-1}-u_{sQ-2}) & 0 \\ 0 & 0 & . & 1 & \exp(j(u_{sQ}-u_{sQ-1}) \end{bmatrix}$$

The AP architecture........................

A Kalman tracker of the source trajectory can be added to further improve the resulting performance



The processing in every branch reduces to a blocking of the rest of the sources followed by a phased array scanning as corresponds to the ML estimate for a single source.

These two steps can be done simultaneously by a single beamforming that steering the desired, nulls out the rest of the sources.

Thus, the beamformer scanning for source s has to hold the following constrain:

$$\underline{A}^H \cdot \left[\underline{S}, \underline{S}_1^{(m)}, .., \underline{S}_{s-1}^{(m)}, \underline{S}_{s+1}^{(m)}, .., \underline{S}_{NS}^{(m)} \right] = [1,0,..,0]$$

In addition, it has to shown minimum response to the white noise, i.e. has to be minimum norm.

$$\left. \underline{A}^H \underline{A} \right|_{MIN}$$

The new estimate of the DOA for source S, results from the absolute maxima of

$$S_s^{(m+1)} = \max_{\underline{S}} \underline{A}^H . \underline{\underline{R}}_x . \underline{A}$$

In general: For s=1:NS

$$\underline{\underline{C}}_s = \left[\underline{S}, \underline{S}_1^{(m)}, \ldots, \underline{S}_{s-1}^{(m)}, \underline{S}_{s+1}^{(m)}, \ldots, \underline{S}_{NS}^{(m)}\right] \qquad \underline{1}^T = [1\,0\,..\,0]$$

$$\underline{A} = \underline{\underline{C}}_s . \left(\underline{\underline{C}}_s^H . \underline{\underline{C}}_s\right)^{-1} . \underline{1}$$

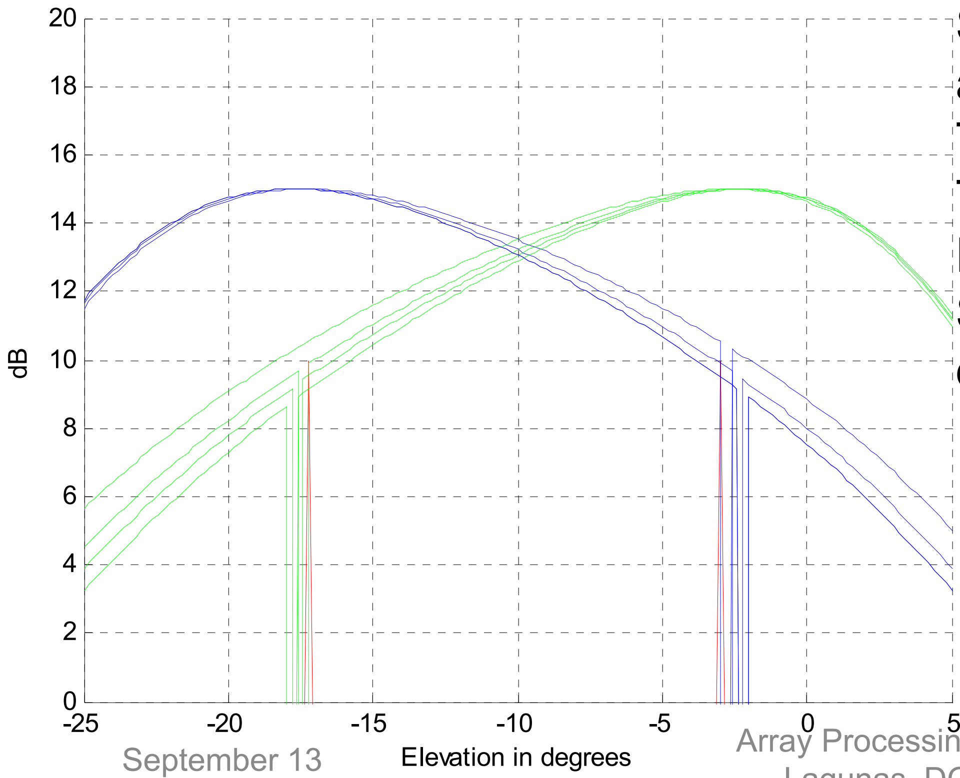
$$\underline{S}_s^{(m+1)} = \max_{\underline{S}} \left(\frac{\underline{A}^H . \underline{\underline{R}}_x . \underline{A}}{\underline{A}^H . \underline{A}} = \frac{\underline{1}^H . \left(\underline{\underline{C}}_s^H . \underline{\underline{C}}_s\right)^{-1} . \underline{\underline{C}}_s^H . \underline{\underline{R}}_x . \underline{\underline{C}}_s . \left(\underline{\underline{C}}_s^H . \underline{\underline{C}}_s\right)^{-1} . \underline{1}}{\underline{1}^H . \left(\underline{\underline{C}}_s^H . \underline{\underline{C}}_s\right)^{-1} . \underline{1}} \right)$$

end

Note that we divide by the noise bandwidth of the scanning beamformer to form the DOA estimate

Example of AP performance

Two sources. ULA array half wavelength.

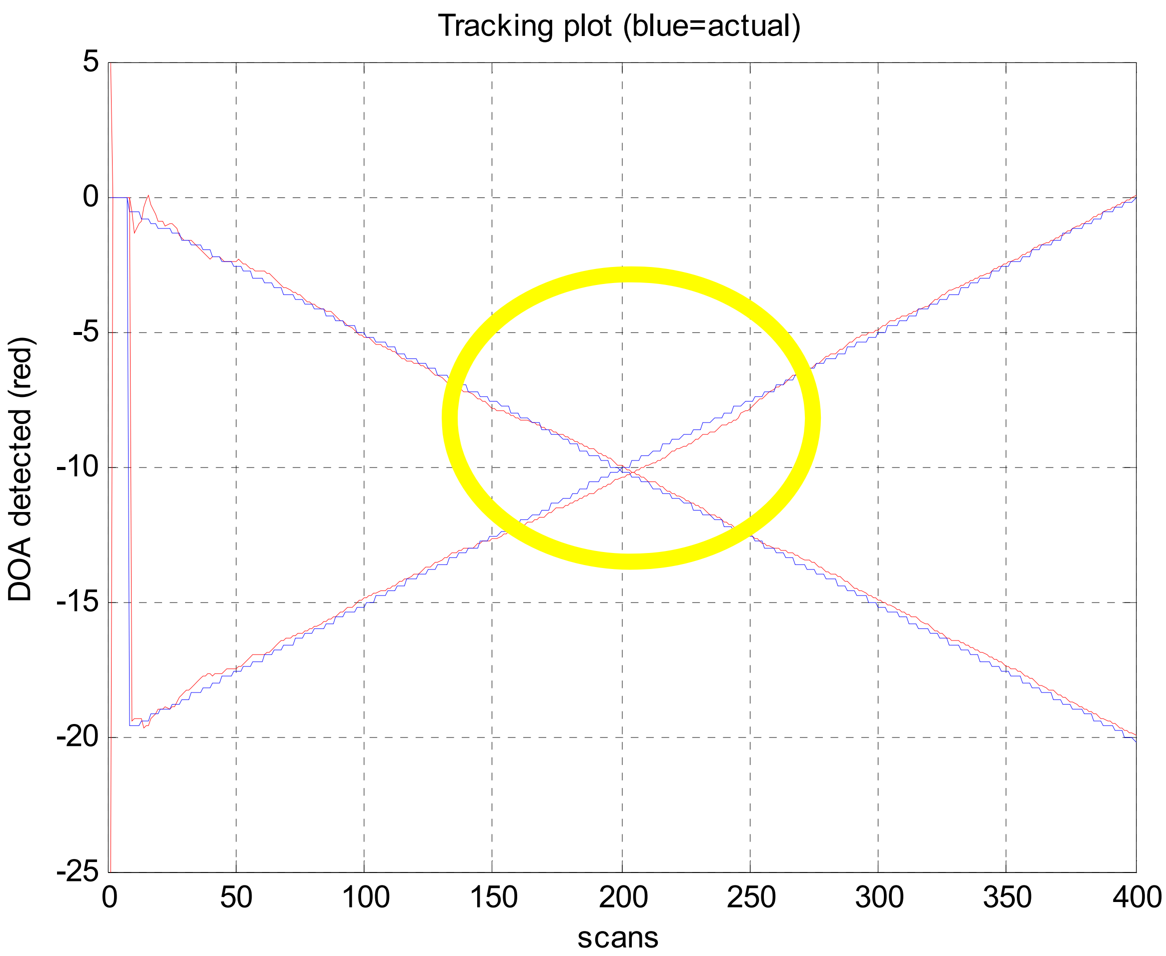


The number of snapshots is ...4000
WAR source 1 is coherent with source 2
The number of sources is...2
Source #1--> 10dB 0 º elevation 0 º azimuth
The velocity in elevation is -0.005 º/snap
Source #2--> 10dB -20 º elevation 0 º azimuth
The velocity in elevation is 0.005 º/snap
The number of sensors is......8
Elevation Field of view from -25 up to 5
Scanning precission in elevation of 0.2 degrees

Snapshot # 4000

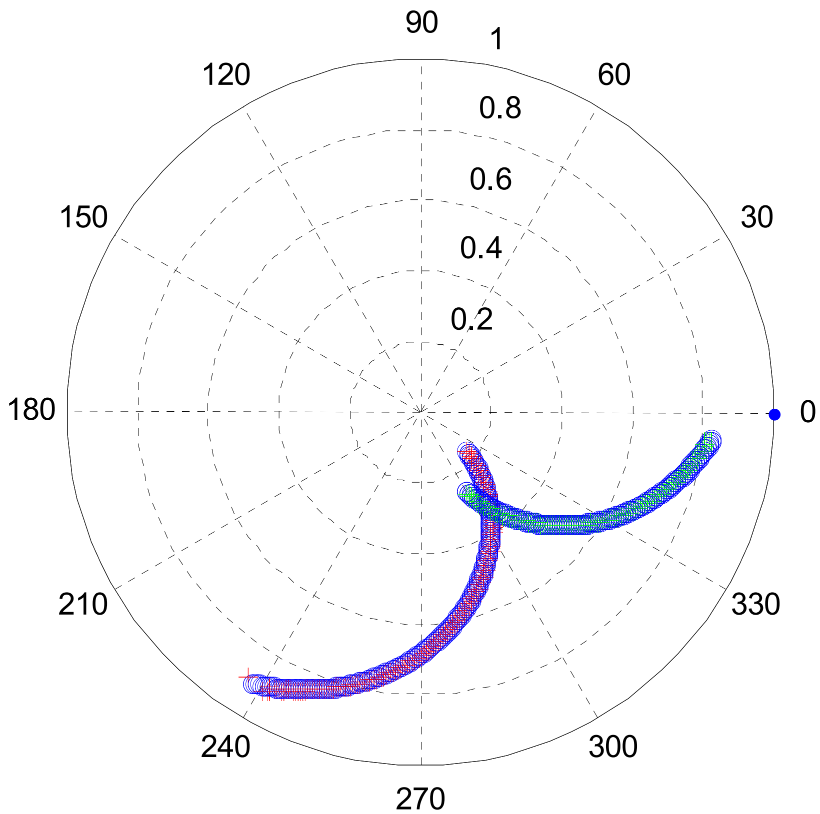
proc.#1 doa -20º radial vel. -0.0049529

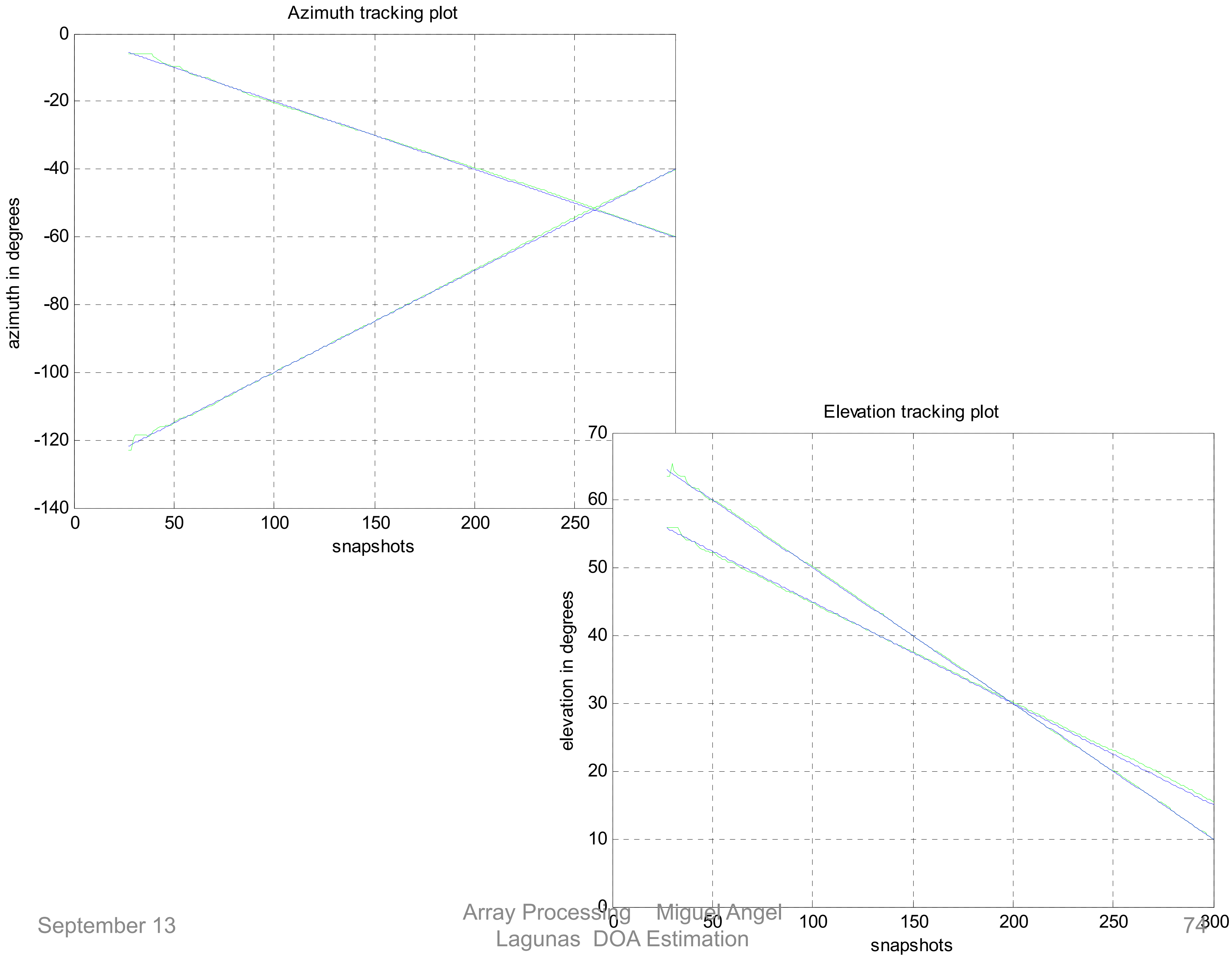
proc.#2 doa 0º radial vel. 0.0050821



The number of snapshots is...3000
 WAR source 1 is coherent with source 2
The number of sources is...2
Source #1--> 20dB 60 º elevation 0 º azimuth
   Elevation velocity -0.015 º/snapshot
   Azimuth velocity -0.02 º/snapshot
Source #2--> 20dB 70 º elevation -130 º azimuth
   Elevation velocity -0.02 º/snapshot
   Azimuth velocity 0.03 º/snapshot
The number of sensors is...13
Elevation Field of view from 5 up to 75
Scanning precission in elevation of 0.2 degrees

slowness/azimuth plot, actual blue, estimated green



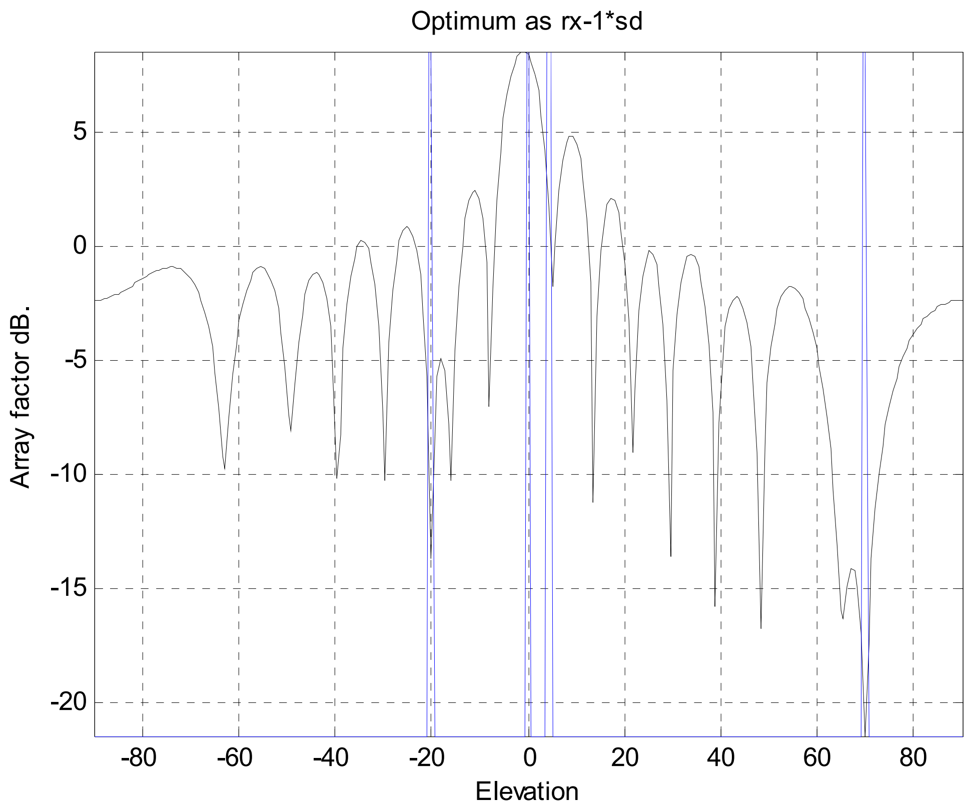

Azimuth tracking plot
azimuth in degrees
snapshots
0
-20
-40
-60
-80
-100
-120
-140
0
50
100
150
200
250
Elevation tracking plot
elevation in degrees
snapshots
70
60
50
40
30
20
10
0
0
50
100
150
200
250
300

AP for Beamforming

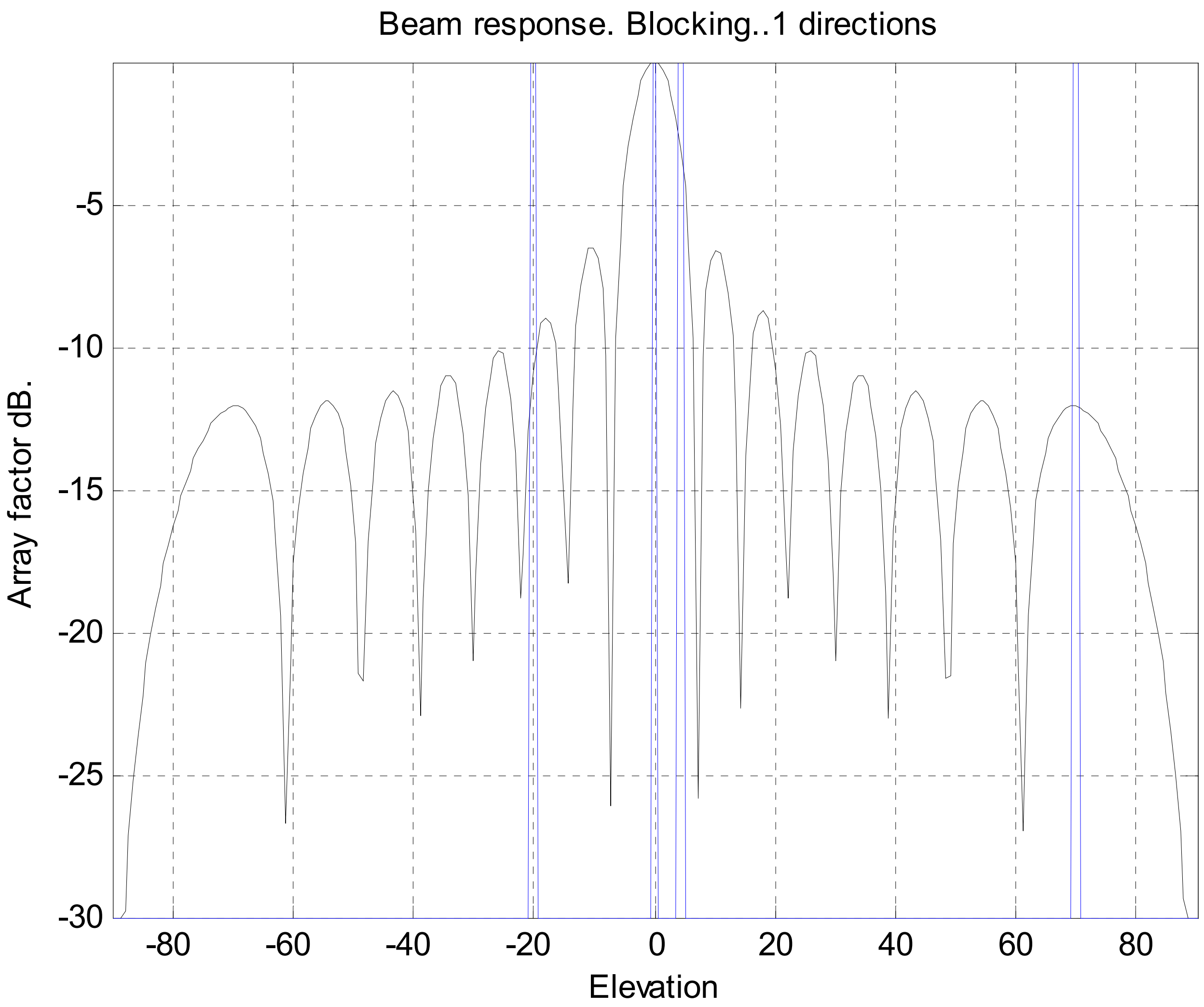
Our goal: A beamforming steeresd to some angle (0º)
With perfect nulling of unknown coherent or uncoherent interferences

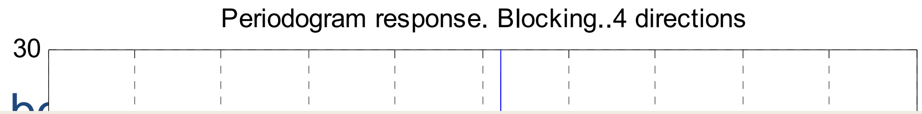
Initial beamforming: Phased Array steered at the desired

The number of snapshots is...5000
WAR source 1 is coherent with source 4
The number of sources is...4
Source #1--> 10dB 0 º elevation
Source #2--> 20dB 70 º elevation
Source #3--> 20dB -20 º elevation
Source #4--> 30dB 5 º elevation
The number of sensors is...16
Elevation scanning data
Elevation of the desired 0º
quiescent response

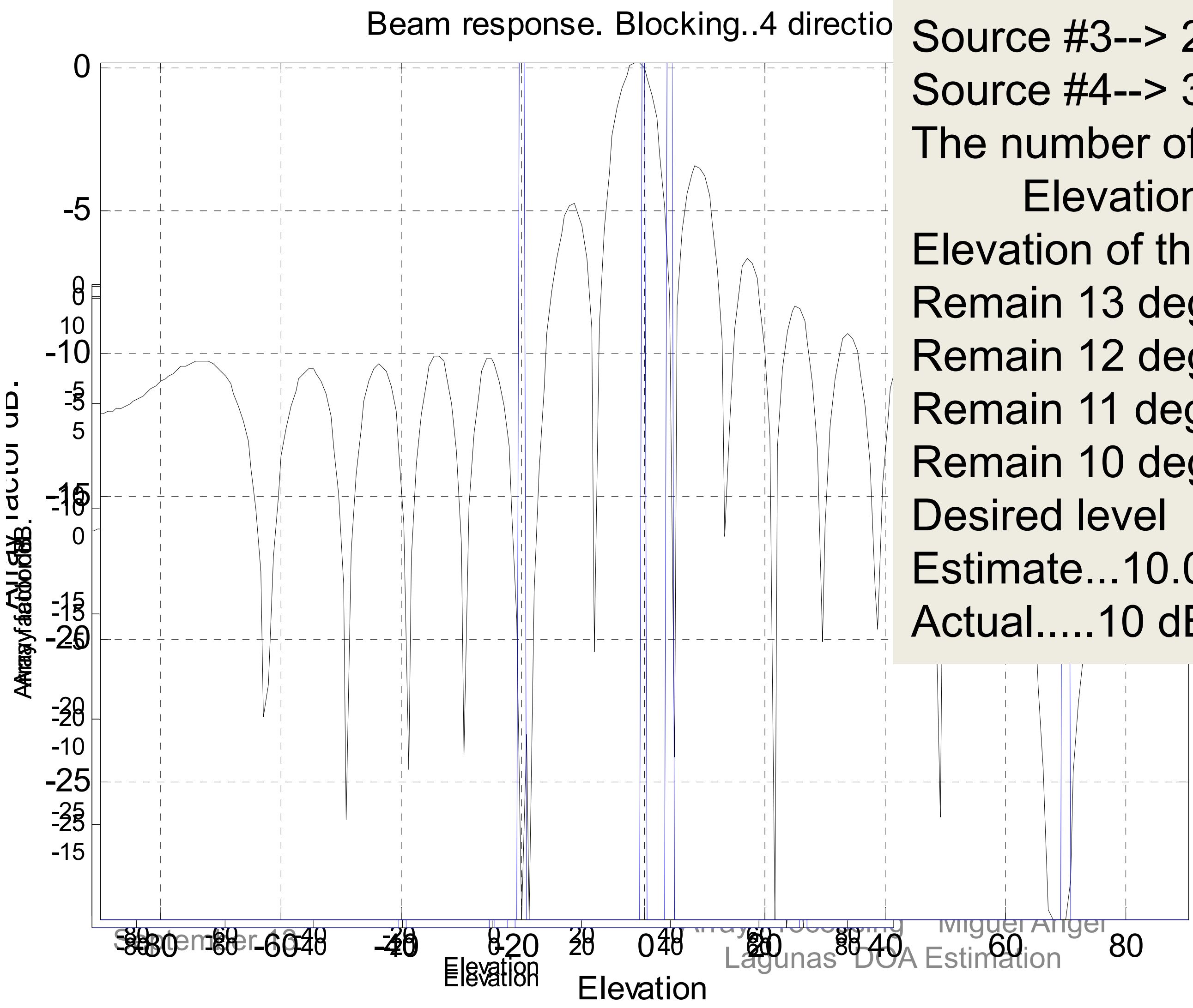

Optimum as rx-1*sd
Array factor dB.
Elevation
5
0
-5
-10
-15
-20
-80
-60
-40
-20
0
20
40
60
80

Beamformer steered to the desired blocking s1





Scan blocking desired and directions b
The design beamformer steered to t
minimum norm.
CONTINUE until flat scanning



The number of snapshots is...5000
The number of sources is...4
Source #1--> 10dB 0 º elevation
Source #2--> 20dB 70 º elevation
Source #3--> 20dB -20 º elevation
Source #4--> 30dB 5 º elevation
The number of sensors is...16

Elevation scanning data

Elevation of the desired 0º
Remain 13 degrees of freedom
Remain 12 degrees of freedom
Remain 11 degrees of freedom
Remain 10 degrees of freedom
Desired level
Estimate...10.0591 dB.
Actual.....10 dB.

WIDE-BAND DOA ESTIMATION

The estimate depending on DOAs and frequency is obtained by defining a steering vector for the wideband beamformer (Q sensor and NT tap delays).

$$\boxed{\underline{S}} = \left[\underline{S}^T, \underline{S}^T \exp\left(j2\pi fT\right), \ldots, \underline{S}^T \exp\left(j2\pi(N-1)\, fT\right)\right]^T$$

where

$$\underline{S} = \exp\left(j2\pi . f \frac{\sin(\theta)}{c} \left(\underline{d} \otimes \cos\left(\varphi - \underline{\varphi}\right)\right)\right) . \exp\left(- j2\pi mfT\right)$$

The beamformer output is:

$$y(n) = \underline{a}^H . \boxed{\left[\underline{X}_n^T \, \underline{X}_{n-1}^T \ldots . \underline{X}_{n-N+1}^T\right]^T}$$

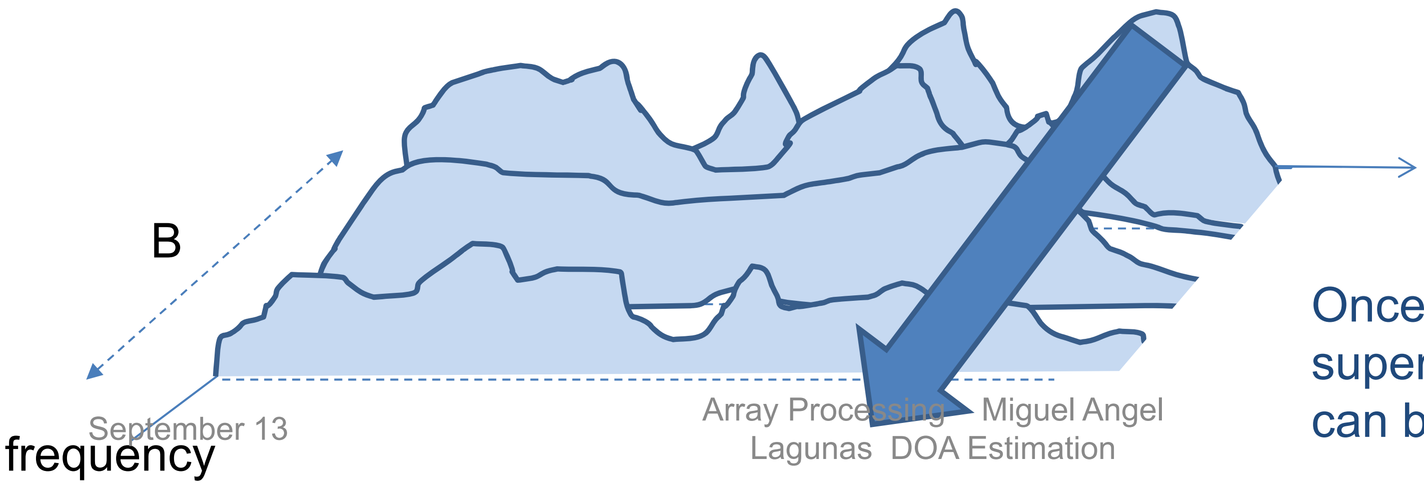
Thus, replacing the steering and the new sanpshot, all the procedures previously can be extended to the wideband case

Focusing on Wideband estimation

All the wideband estimates needs of focusing the source. The reason is that, given a source, on position (ϴ,φ), that impinges the aperture in a bandwidth B, in order to properly detecting the source we need to obtain the power that the source produces at the array output. This power is computed as the sum of the powers measured on the bandwidt B.

$$\int_B \Phi(\theta,\varphi,f)\,df$$

This averaging of all the map is also known as focusing (resume in a single plot) the power per angle solid



Once power is focalized, super-resolution methods can be applied.

Focusing for ULA

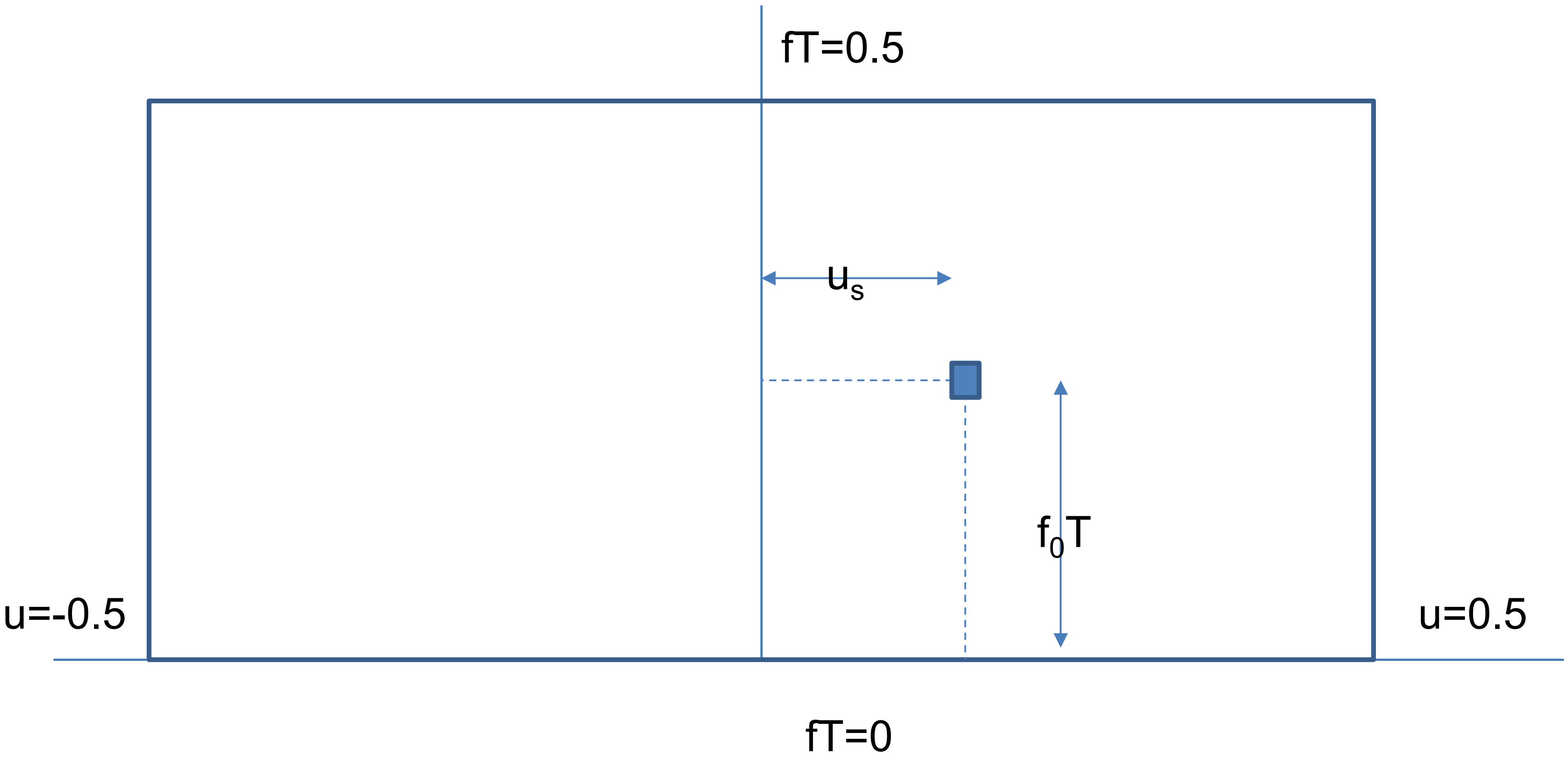
For an ULA array of Q sensors and M delay lags, the output of a wideband phased array will be:

$$X(u, f, n) = \sum_{m=1}^{M} \sum_{q=1}^{Q} \underline{\underline{X}}_{n} . \exp(-j2\pi fmT) . \exp(-j2\pi q.u)$$

Thus, in this case, the output is like a 2D-DFT one in the spatial domain with frequency u and the other in the time domain with frequency fT

Let us imagine that we have in the scenario a single source located at elevation Θ_s , located at f_0 with bandwidth almost zero.

The 2D-DFT will show a delta function locates at spatial frequency equal to u_s and frequency f_0T. The next figure depicts the situation in the spatial time frequency plot.



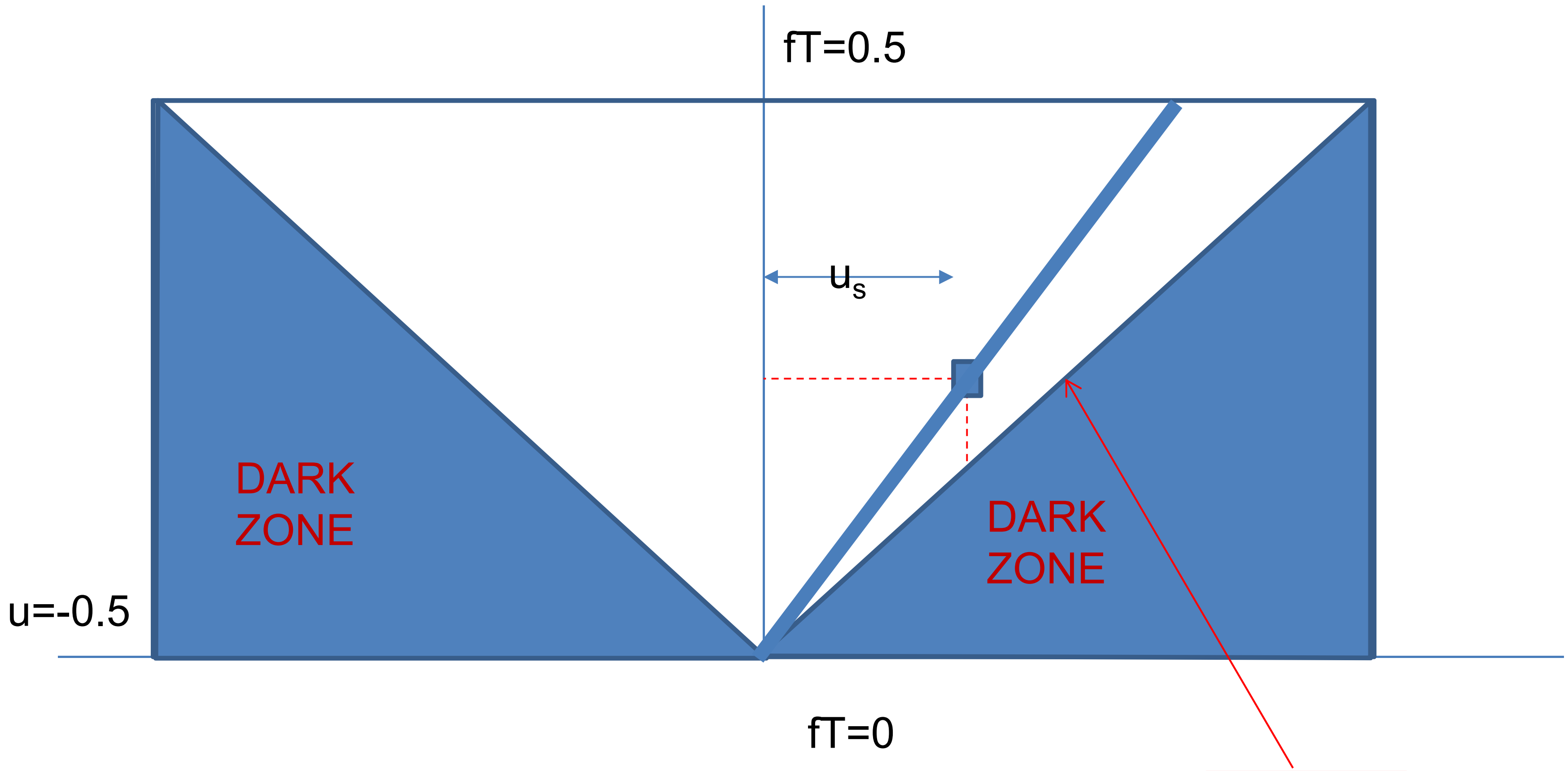
But, THERE IS A BASIC DIFFERENCE between wideband DOA detection and traditional 2D spectral estimation namely THE TWO FREQUENCIES ARE COUPLED

$$u = \frac{f.d}{c}.\mathrm{sen}(\theta)$$

and

$$f = \frac{c}{d.sen(\theta)}.u \geq [\theta = 90^{o}] = \frac{c}{d}.u$$

$$u \leq \frac{f.d}{c}$$



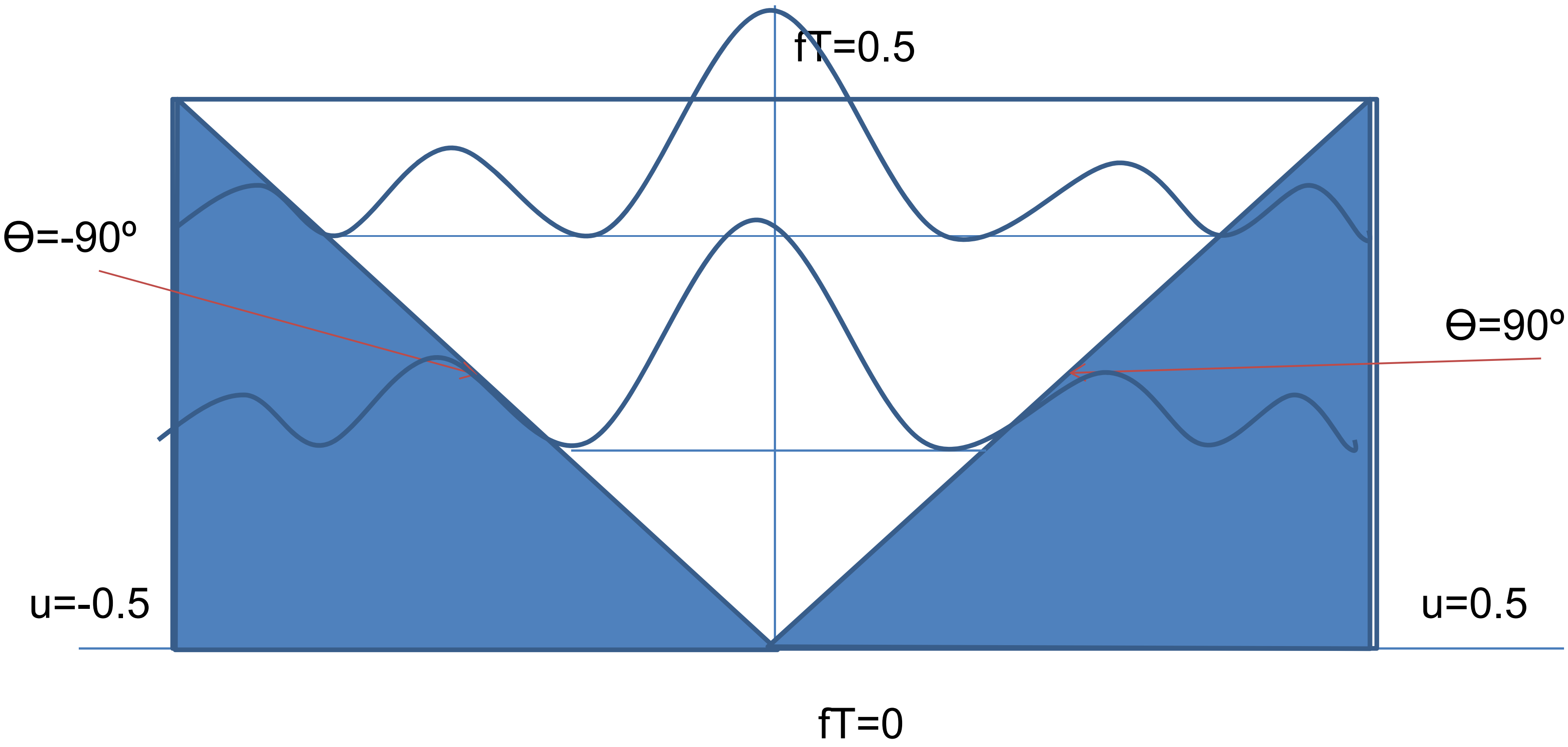
$$u \le \frac{f.d}{c}$$

When the source extents its bandwidth the u frequency moves along a line

$$u = \left(dfT / cT\right)\sin(\theta)$$

In consequence, the power have to be focused along lines with slop equal to:

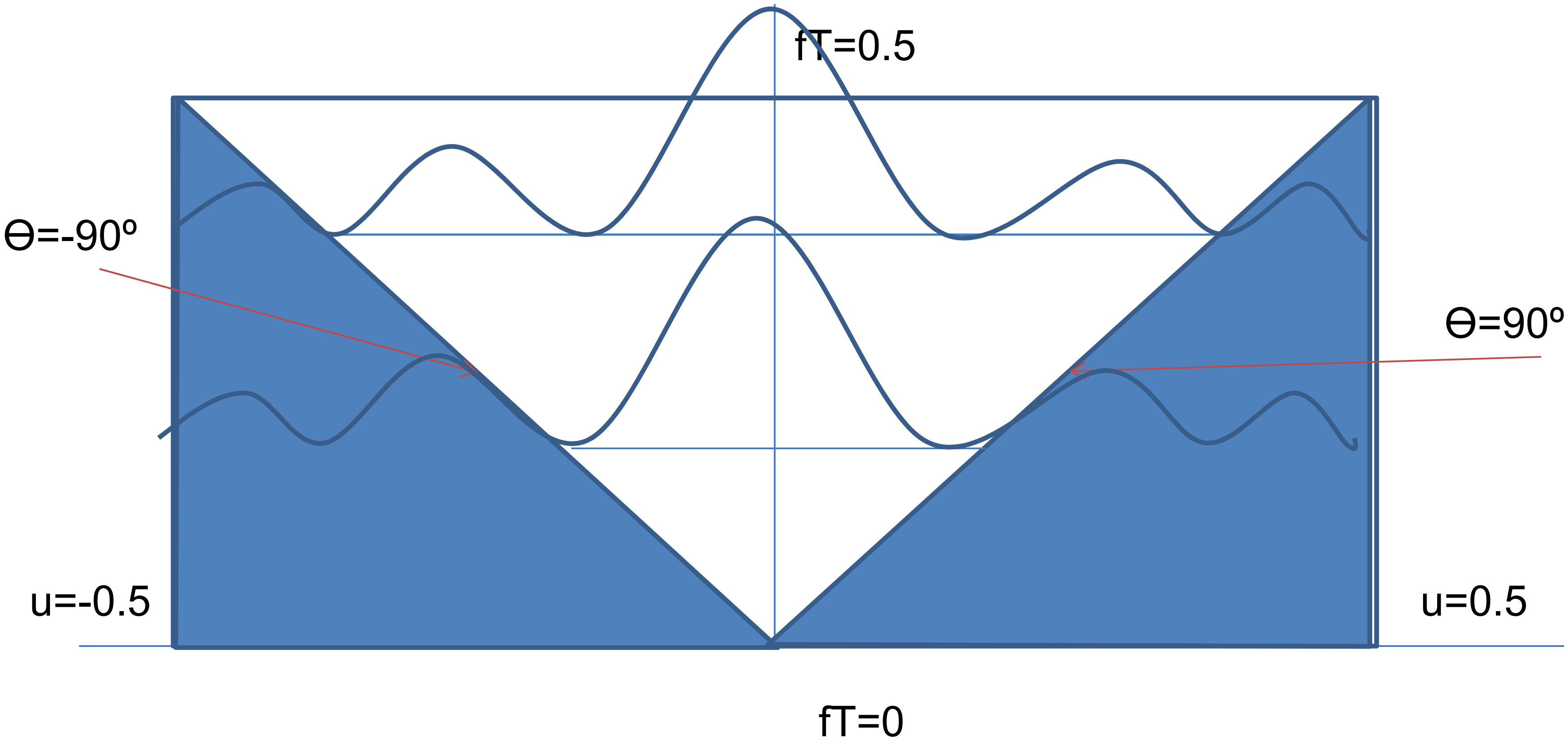
$$\frac{d}{c}\sin\left(\theta\right)$$

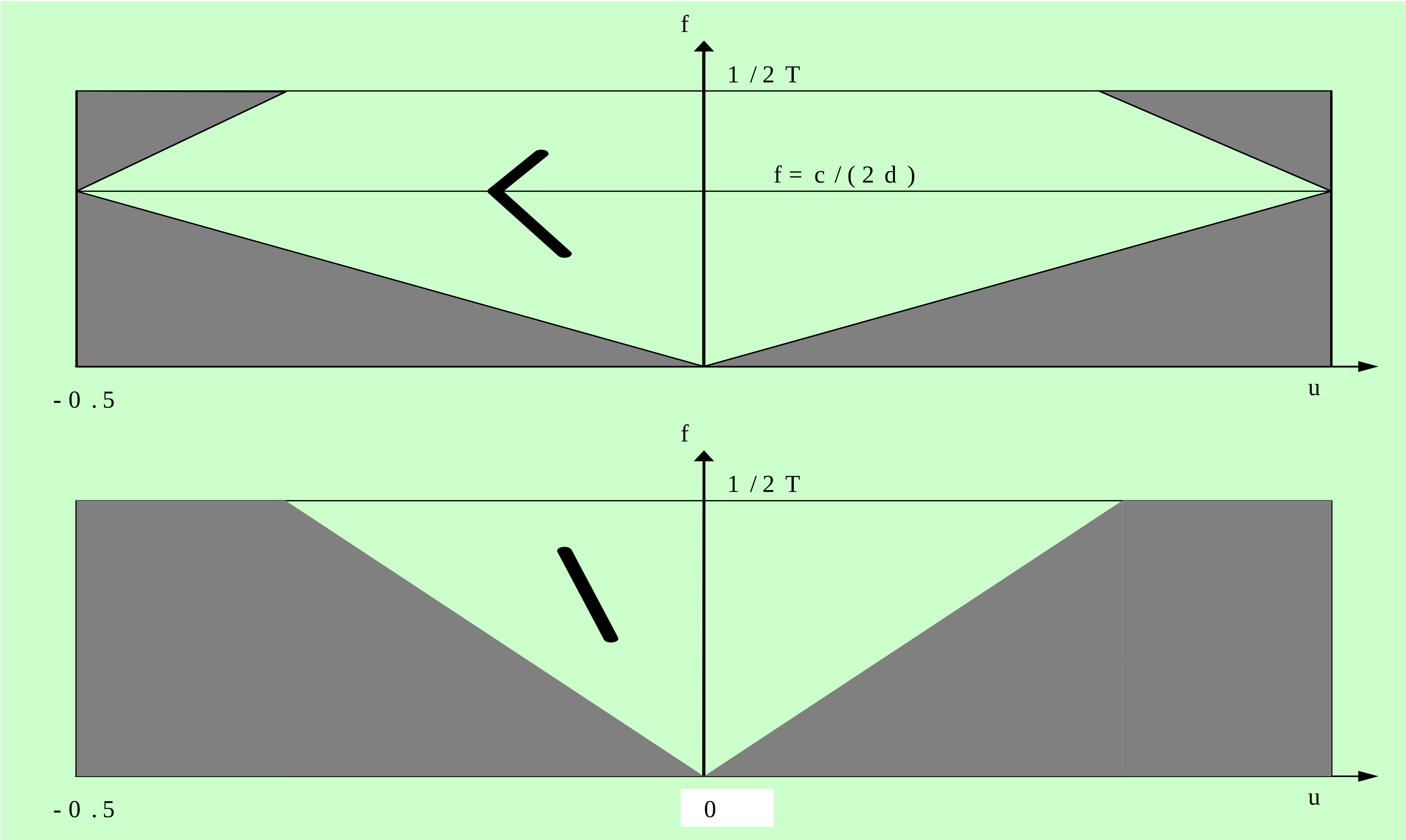


For a central frequency f_c, the sampling frequency as $T=1/2f_{max}$. Selecting the sensor separation d as $d=\lambda_c/2$, we have:

$$\frac{d}{cT}\sin(\theta)=\frac{2f_{max}\lambda_c}{2c}\sin(\theta)=\left(\frac{f_{max}}{f_c}\right)\sin(\theta)$$

Note that for a given response, the resolution is always lower at low frequencies than at high frequencies





Top: Corect choice of separation d in order to have maximum field of view. Also for sources at the maximum vertical focussing is a good approsimation. Botton Wrong choice of the interelement separation (large d)

Focusing for super-resolution

Let us imagine that we have the array covariance matrix at a given frequency f as:

$$\underline{\underline{R}}(f)=\underline{\underline{S}}(f)\underline{\underline{P}}(f).\underline{\underline{S}}^{H}(f)+\sigma^{2}(f)\underline{\underline{I}}$$

Focusing this matrix at frequency f_c implies to find out a transformation such that:

$$\underline{\underline{T}}\,\underline{\underline{R}}(f)\underline{\underline{T}}^{H}=\underline{\underline{S}}(f_c)\underline{\underline{P}}(f).\underline{\underline{S}}^{H}(f_c)+\sigma^{2}(f)\underline{\underline{I}}$$

With this, summing up all the matrix the power (also the noise) is focused in a single matrix. Using Music NMLM or any other super-resolution method will provide accurate source location

$$\sum_{f}\underline{\underline{T}}\,\underline{\underline{R}}(f)\underline{\underline{T}}^{H}=\underline{\underline{S}}(f_c)\left[\sum_{f}\underline{\underline{P}}(f)\right].\underline{\underline{S}}^{H}(f_c)+\left(\sum_{f}\sigma^{2}(f)\right)\underline{\underline{I}}$$

Back to the design of the proper transformation, we can see that there are two design conditions:

$$\underline{\underline{T}}\,\underline{\underline{R}}(f)\underline{\underline{T}}^H = \underline{\underline{S}}(f_c)\underline{\underline{P}}(f)\underline{\underline{S}}^H(f_c) + \sigma^2(f)\underline{\underline{I}}$$

$$\underline{\underline{T}}\,\underline{\underline{S}}(f) = \underline{\underline{S}}(f_c) \qquad\qquad \underline{\underline{T}}\,\underline{\underline{T}}^H = \underline{\underline{I}}$$

$$or \quad \underline{\underline{T}}\,\underline{\underline{S}}_f = \underline{\underline{S}}_{f_c}$$

Clearly using the pseudo-inverse is not a valid solution

$$\underline{\underline{T}} = \underline{\underline{S}}_{f_c}\left(\underline{\underline{S}}_f^H \underline{\underline{S}}_f\right)^{-1}\underline{\underline{S}}_f^H \qquad \text{since} \quad \underline{\underline{T}}\,\underline{\underline{T}}^H = \underline{\underline{S}}_{f_c}\left(\underline{\underline{S}}_f^H \underline{\underline{S}}_f\right)^{-1}\underline{\underline{S}}_{f_c}^H \neq \underline{\underline{I}}$$

We need to relax the first constrain which is better an more easy than to relax the white spatial noise one.

The design will be:

$$\left| \underline{\underline{T}}\,\underline{\underline{S}}_f - \underline{\underline{S}}_{f_c} \right|_F^2 \quad s.t. \quad \underline{\underline{T}}\,\underline{\underline{T}}^H = \underline{\underline{I}}$$

Taking derivatives

$$\underline{\underline{T}}\left(\underline{\underline{S}}_f \underline{\underline{S}}_f^H - \underline{\underline{L}}\right) = \underline{\underline{S}}_{f_c} \underline{\underline{S}}_f^H$$

With the svd of the DOAs matrixes. Note
That for matrix L this merely instrumental
Since L is unknown.

$$\underline{\underline{S}}_f \underline{\underline{S}}_f^H = \underline{\underline{U}}\,\underline{\underline{D}}_f \underline{\underline{U}}^H$$

$$\underline{\underline{S}}_{f_c} \underline{\underline{S}}_f^H = \underline{\underline{V}}\,\underline{\underline{D}}_{ff_c} \underline{\underline{U}}^H$$

then

$$\underline{\underline{L}} = \underline{\underline{U}}\,\underline{\underline{D}}_l \underline{\underline{U}}^H$$

$$\underline{\underline{T}}\,\underline{\underline{U}}\left(\underline{\underline{D}}_f - \underline{\underline{D}}_l\right)\underline{\underline{U}}^H = \underline{\underline{V}}\,\underline{\underline{D}}_{ff_c} \underline{\underline{U}}^H$$

Clearly, the
solution is:

$$\boxed{\underline{\underline{T}} = \underline{\underline{V}}\,\underline{\underline{U}}^H}$$

EJERCICIO

Una manera diferente para derivar metodos de estimacion de DOA consiste en metodos denominados de “covariance matching”, es decir, de ajuste de la matriz de covarianza observada.
Sea la matriz obtenida a partir de un conjunto de N snapshots, siendo N suficiente como para estabilizar el estimador.

$$\underline{\underline{R}} = \frac{1}{N}\sum_{m=1}^{N} \underline{X}_m \underline{X}_m^H$$

Es claro que el contenido de potencia ξ que proviene de una direccion dada viene dado por

$$\underline{\underline{R}}_c = \xi \underline{S}.\underline{S}^H$$

Asi pues para saber que contenido de potencia tiene nuestra observacion R en una direccion dada, basta encontrar el valor de ξ que mas “asemeja” la observacion R con R_c. Es decir el que minimiza la “diferencia” entre las dos matrices

$$\boxed{P(\underline{S}) = \min_{\xi} f\left(\underline{\underline{R}}, \underline{\underline{R}}_c\right)}$$

PRIMER CASO: f(,) es la norma de Frobenius de la matriz diferencia.

$$\left| \underline{\underline{R}} - \underline{\underline{R}}_c \right|_F = \left| \underline{\underline{R}} - \xi \underline{S} . \underline{S}^H \right|_F$$

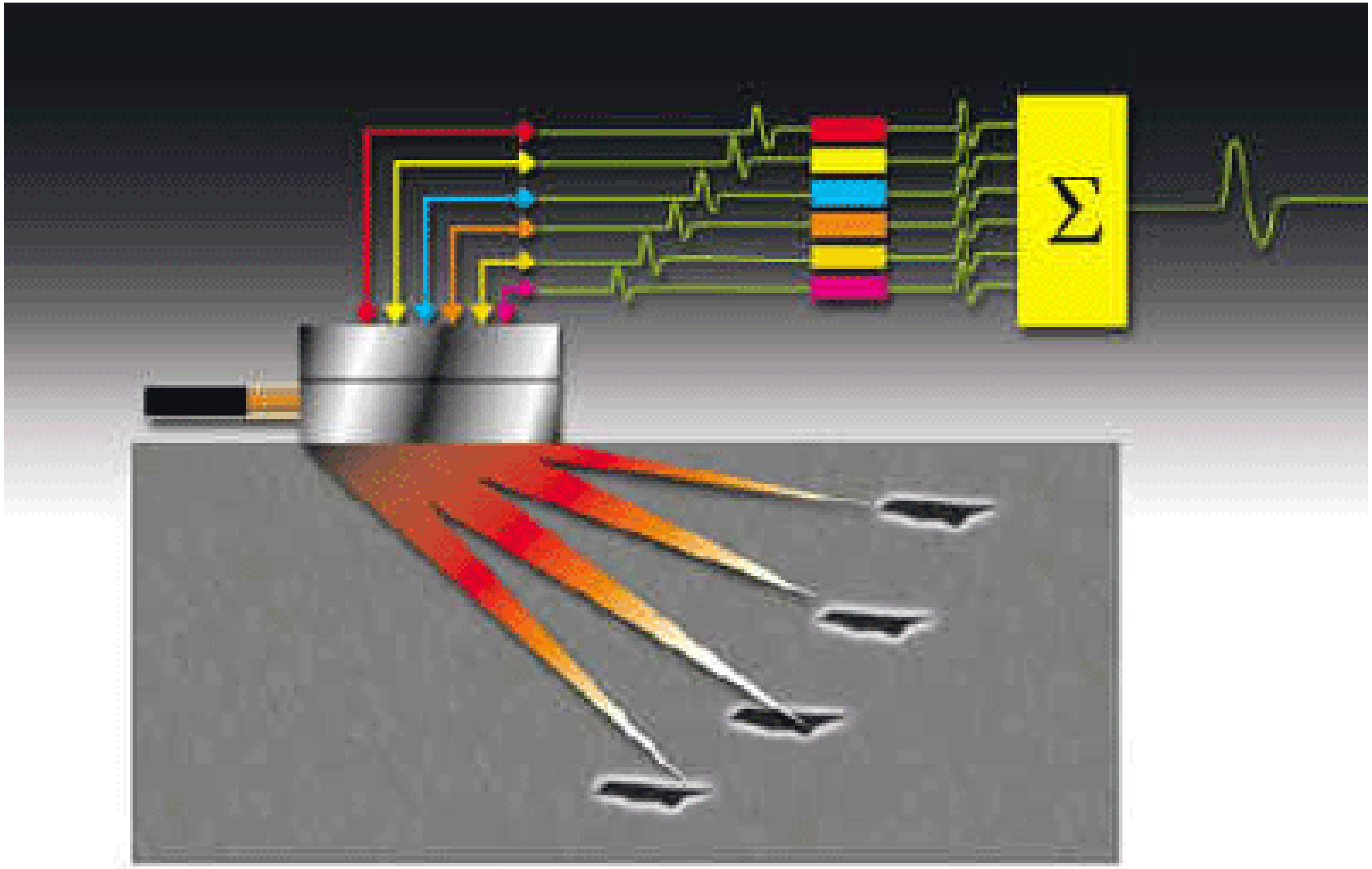
Encontrar el estimador de la potencia, para cada direccion que minimiza esta distancia

Nota: $\left| \underline{\underline{A}} \right|_F = traza\left(\underline{\underline{A}}^H . \underline{\underline{A}} \right)$ $\nabla \left[traza\left(\underline{\underline{A}}^H . \underline{\underline{A}} \right) \right] = \left(\nabla \underline{\underline{A}}^H \right) \underline{\underline{A}}$

SEGUNDO CASO: Si a la observada le restamos la potencia que le llega de una direccion, la maxima potencia que podemos restar seria la que es aquella que no hace la matriz diferencia definida negativa. Esta matriz diferencia tendria al menos un autovalor cero, por lo que la nueva definicion de f(,) para encontrar ξ seria:

$$P(\underline{S}) = \max_{\xi} \left[\lambda_{\min} \left(\underline{\underline{R}} - \xi \underline{S}\, \underline{S}^H \right) = 0 \right]$$

Encontrar el estimador de la potencia, para cada direccion como el valor maximo que provica en la diferencia un autovalor cero.


Σ